Classification of connected holonomy groups for pseudo-Kählerian manifolds of index 2

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Abstract

The problem of classification of connected holonomy groups (equivalently of holonomy algebras) for pseudo-Riemannian manifolds is open. The classification of Riemannian holonomy algebras is a classical result. The classification of Lorentzian holonomy algebras was obtained recently.

In the present paper weakly-irreducible not irreducible subalgebras of $\mathfrak{su}(1,n+1)$ ($n\geq 0$) are classified. Weakly-irreducible not irreducible holonomy algebras of pseudo-Kählerian and special pseudo-Kählerian manifolds are classified. An example of metric for each possible holonomy algebra is given. This gives the classification of holonomy algebras for pseudo-Kählerian manifolds of index 2.

Keywords: pseudo-Kählerian manifold, holonomy group

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Contents

1 Introduction

The classification of connected holonomy groups of Riemannian manifolds is well known. First in 1952 A. Borel and A. Lichnerowicz proved that a Riemannian manifold is locally a product of Riemannian manifolds with irreducible holonomy groups, see [9]. In 1955 M. Berger gave a list of possible connected irreducible holonomy groups of Riemannian manifolds, see [8]. Later, in 1989 R. Bryant constructed metrics for the exceptional groups of this list, see [10].

In the case of pseudo-Riemannian manifolds appears the situation when the holonomy group preserves a degenerate vector subspace of the tangent space. In this situation the Borel-Lichnerowicz theorem does not work. A subgroup $G \subset SO(p,q)$ is called *weakly-irreducible* if it does not preserve any nondegenerate proper subspace of $\mathbb{R}^{p,q}$. The Wu theorem states that *a pseudo-Riemannian manifold is locally a product of pseudo-Riemannian manifolds with weakly-irreducible holonomy groups*, see [22]. This reduses the problem of classification of holonomy groups of pseudo-Riemannian manifolds to the weakly-irreducible case. If a holonomy group is irreducible, then it is weakly-irreducible. In [8] M. Berger gave also a list of possible connected irreducible holonomy groups for pseudo-Riemannian manifolds.

Thus the first problem is to classify weakly-irreducible not irreducible subgroups of $SO(p,q)$. This was completely done only for connected groups in the Lorentzian case, i.e. for the signature $(1,n+1)$, in 1993 by L. Berard Bergery and A. Ikemakhen, who divided connected weakly-irreducible not irreducible subgroups of $SO(1,n+1)$ into 4 types, see [5].

In [12] more geometrical proof of this result was given, see also remark after Theorem 2.3. To each weakly-irreducible not irreducible subgroup of $G \subset SO(1, n+1)$ can be associated a subgroup of $SO(n)$, which is called the orthogonal part of G. Just recently T. Leistner showed that the orthogonal part of a weakly-irreducible not irreducible holonomy group of a Lorentzian manifold must be the holonomy group of a Riemannian manifold, see [18, 19, 20]. In [13] metrics for groups of each type with the given connected holonomy group of a Riemannian manifold as the orthogonal part are constructed. This completes the classification of connected holonomy groups for Lorentzian manifolds.

The next signature to study is $(2, N)$. In 1998 A. Ikemakhen classified connected weakly-irreducible subgroups of $SO(2, N)$ that preserve an isotropic plane and satisfy an additional condition, see [15]. We study connected holonomy groups of pseudo-Kählerian manifolds of signature $(2, 2n+2)$, i.e. holonomy groups contained in $U(1, n+1) \subset SO(2, 2n+2)$. From the Wu theorem it follows that any such group is a product of irreducible holonomy groups of Kählerian manifolds and of the weakly-irreducible holonomy group of a pseudo-Kählerian manifold of signature $(2, 2k+2)$.

Let $\mathbb{R}^{2,2n+2}$ be a $2n+4$-dimensional real vector space endowed with a complex structure J and with a J-invariant metric η of signature $(2, 2n+2)$ $(n \geq 0)$. In **Section 2** we classify (up to conjugacy) all connected subgroups of $SU(1, n+1)$ that act weakly-irreducibly and not irreducibly on $\mathbb{R}^{2,2n+2}$, that is equivalent to the classification of the corresponding subalgebras of $\mathfrak{su}(1, n+1)$. Any such subgroup preserves a 2-dimensional isotropic J-invariant subspace of $\mathbb{R}^{2,2n+2}$. We use a generalization of the method from [12].

As the first case, we consider all subalgebras of $\mathfrak{su}(1, 1)$ that preserve a 2-dimensional isotropic J-invariant subspace of $\mathbb{R}^{2,2}$ and show which of these subalgebras are weakly-irreducible.

Then we consider the case $n \geq 1$. We denote by $\mathbb{C}^{1,n+1}$ the $n+2$-dimensional complex vector space given by $(\mathbb{R}^{2,2n+2}, J, \eta)$. Let g be the pseudo-Hermitian metric on $\mathbb{C}^{1,n+1}$ of signature $(1, n+1)$ corresponding to η. If a subgroup $G \subset U(1, n+1)$ acts weakly-irreducibly on $\mathbb{R}^{2,2n+2}$, then G acts weakly-irreducibly on $\mathbb{C}^{1,n+1}$, i.e. does not preserve any proper g-non-degenerate complex vector subspace.

We consider the boundary $\partial \mathbf{H}^{n+1}_{\mathbb{C}}$ of the complex hyperbolic space $\mathbf{H}^{n+1}_{\mathbb{C}}$ and identify $\partial \mathbf{H}^{n+1}_{\mathbb{C}}$ with the $2n+1$-dimensional sphere S^{2n+1}. We fix a complex isotropic line $l \subset \mathbb{C}^{1,n+1}$ and denote by $U(1, n+1)_l \subset U(1, n+1)$ the connected Lie subgroup that preserves the line l. Any connected subgroup $G \subset U(1, n+1)$ that acts on $\mathbb{C}^{1,n+1}$ weakly-irreducibly

and not irreducibly is conjugated to a subgroup of $U(1,n+1)_l$.

We identify the set $\partial\mathbf{H}^{n+1}_{\mathbb{C}}\backslash\{l\}=S^{2n+1}\backslash\{point\}$ with the Heisenberg space $\mathcal{H}_n=\mathbb{C}^n\oplus\mathbb{R}$. Any element $f\in U(1,n+1)_l$ induces a transformation $\Gamma(f)$ of $\mathcal{H}_n$, moreover, $\Gamma(f)\in\mathrm{Sim}\,\mathcal{H}_n$, where $\mathrm{Sim}\,\mathcal{H}_n$ is the group of the Heisenberg similarity transformations of $\mathcal{H}_n$. We show that $\Gamma:U(1,n+1)_l\to\mathrm{Sim}\,\mathcal{H}_n$ is a surjective Lie group homomorphism with the kernel $\mathbb{T}$, where $\mathbb{T}$ is the 1-dimensional subgroup generated by the complex structure $J\in U(1,n+1)_l$. In particular, $\mathbb{T}$ is the center of $U(1,n+1)_l$. Let $SU(1,n+1)_l=U(1,n+1)_l\cap SU(1,n+1)$. Then $U(1,n+1)_l=SU(1,n+1)_l\cdot\mathbb{T}$ and the restriction

$$\Gamma|_{SU(1,n+1)_l}:SU(1,n+1)_l\to\mathrm{Sim}\,\mathcal{H}_n$$

is a Lie group isomorphism.

We consider the projection $\pi:\mathrm{Sim}\,\mathcal{H}_n\to\mathrm{Sim}\,\mathbb{C}^n$, where $\mathrm{Sim}\,\mathbb{C}^n$ is the group of similarity transformations of $\mathbb{C}^n$. The homomorphism π is surjective and its kernel is 1-dimensional.

We prove that *if a subgroup $G\subset U(1,n+1)_l$ acts weakly-irreducibly on $\mathbb{C}^{1,n+1}$, then*

(1) *the subgroup $\pi(\Gamma(G))\subset\mathrm{Sim}\,\mathbb{C}^n$ does not preserve any proper complex affine subspace of $\mathbb{C}^n$;*

(2) *if $\pi(\Gamma(G))\subset\mathrm{Sim}\,\mathbb{C}^n$ preserves a proper non-complex affine subspace $L\subset\mathbb{C}^n$, then the minimal complex affine subspace of $\mathbb{C}^n$ containing L is $\mathbb{C}^n$.*

This is the key statement for our classification.

Since we are interested in connected Lie groups, it is enough to classify the corresponding Lie algebras. The classification is done in the following way:

• First we describe non-complex vector subspaces $L\subset\mathbb{C}^n$ with $\mathrm{span}_{\mathbb{C}}L=\mathbb{C}^n$ (it is enough to consider only vector subspaces, since we do the classification up to conjugacy). Any such non-complex vector subspace has the form $L=\mathbb{C}^m\oplus\mathbb{R}^{n-m}$, where $0\leq m\leq n$. Here we have 3 types of subspaces: 1) $m=0$ (L is a real form of $\mathbb{C}^n$); 2) $0<m<n$; 3) $m=n$ ($L=\mathbb{C}^n$).

• We describe the Lie algebras $\mathfrak{f}$ of the connected Lie subgroups $F\subset\mathrm{Sim}\,\mathbb{C}^n$ preserving L. Without loss of generality, we can assume that each Lie group F does not preserve any proper affine subspace of L. This means that F acts irreducibly on L. By a theorem of D.V. Alekseevsky [2, 3], F acts transitively on L. In our recent paper [12] we divided transitive similarity transformation groups of Euclidean spaces into 4 types. Here we unify two of the types. The group F is contained in $(\mathbb{R}^+\times SO(L)\times SO(L^{\perp_\eta}))\ltimes L$,

where $\mathbb{R}^+$ is the group of real dilations of $\mathbb{C}^n$ about the origin and L is the group of all translations in $\mathbb{C}^n$ by vectors of L. In general situation we know only the projection of F on $\mathrm{Sim}\, L = (\mathbb{R}^+ \times SO(L)) \ltimes L$, but in our case the projection of F on $SO(L) \times SO(L^{\perp_\eta})$ is also contained in $U(n)$ and we know the full information about F. On this step we obtain 9 types of Lie algebras.

• Then we describe subalgebras $\mathfrak{a} \subset \mathrm{LA}(\mathrm{Sim}\, \mathcal{H}_n)$ with $\pi(\mathfrak{a}) = \mathfrak{f}$. For each $\mathfrak{f}$ we have 2 possibilities: $\mathfrak{a} = \mathfrak{f} + \ker \pi$ or $\mathfrak{a} = \{x + \zeta(x) | x \in \mathfrak{f}\}$, where $\zeta : \mathfrak{f} \to \ker \pi$ is a linear map. Using the isomorphism $(\Gamma|_{\mathfrak{su}(1,n+1)_l})^{-1}$ we obtain a list of subalgebras $\mathfrak{g} \subset \mathfrak{su}(1, n+1)_l$. This gives us 12 types of Lie algebras.

• Finally we check which of the obtained subalgebras of $\mathfrak{su}(1, n+1)_l \subset \mathfrak{so}(2, 2n+2)$ are weakly-irreducible. It turns out that some of the types contain Lie algebras that are not weakly-irreducible. Giving new definitions to these types we obtain 11 types of weakly-irreducible Lie algebras. Unifying some of the types we obtain 7 types of weakly-irreducible subalgebras of $\mathfrak{su}(1, n+1)_l \subset \mathfrak{so}(2, 2n+2)$.

The result can be stated as follows.

Let $n = 0$. The Lie algebra $\mathfrak{su}(1,1)_l$ is 2-dimensional nilpotent, we have $\mathfrak{su}(1,1)_l = \mathbb{R} \ltimes \mathbb{R}$ and $[(a, 0), (0, c)] = (0, 2ac)$. There are 2 weakly-irreducible subalgebras of $\mathfrak{su}(1,1)_l$: $\{(0, c) | c \in \mathbb{R}\}$ and the whole $\mathfrak{su}(1,1)_l$.

Let $n > 0$. For the Lie algebra $\mathfrak{su}(1, n+1)_l$ we have the Iwasawa decomposition

$$\mathfrak{su}(1, n+1)_l = (\mathbb{R} \oplus \mathfrak{u}(n)) \ltimes \mathrm{LA}\, \mathcal{H}_n,$$

where $\mathrm{LA}\, \mathcal{H}_n = \mathbb{C}^n \ltimes \mathbb{R}$ is the Lie algebra of the Lie group $\mathcal{H}_n$ of the Heisenberg translations of the Heisenberg space $\mathcal{H}_n$.

Let $0 \leq m \leq n$ be an integer. Consider the decomposition $\mathbb{C}^n = \mathbb{C}^m \oplus \mathbb{C}^{n-m}$. Let $\mathfrak{h} \subset \mathfrak{u}(m) \oplus \mathfrak{sod}(n-m)$ be a subalgebra, here $\mathfrak{sod}(n-m) = \left\{ \left(\begin{smallmatrix} B & 0 \\ 0 & B \end{smallmatrix} \right) \middle| B \in \mathfrak{so}(n-m) \right\} \subset \mathfrak{su}(n-m)$. The Lie algebras of one of the types of weakly-irreducible subalgebras of $\mathfrak{su}(1, n+1)_l \subset \mathfrak{so}(2, 2n+2)$ have the form

$$\mathfrak{g}^{m,\mathfrak{h},\mathcal{A}^1} = (\mathbb{R} \oplus \mathfrak{h}) \ltimes ((\mathbb{C}^m \oplus \mathbb{R}^{n-m}) \ltimes \mathbb{R}),$$

where $\mathbb{R}^{n-m} \subset \mathbb{C}^{n-m}$ is a real form. The other types of weakly-irreducible subalgebras of $\mathfrak{su}(1, n+1)_l \subset \mathfrak{so}(2, 2n+2)$ can be obtained from this one using some twisting and they have the following forms:

$\mathfrak{g}^{m,\mathfrak{h},\varphi}= \{\varphi(A)+A|A\in\mathfrak{h}\}\ltimes((\mathbb{C}^m\oplus\mathbb{R}^{n-m})\ltimes\mathbb{R})$,

where $\varphi:\mathfrak{h}\to\mathbb{R}$ is a linear map with $\varphi|_{\mathfrak{h}'}=0$;

$\mathfrak{g}^{n,\mathfrak{h},\psi,k,l}= \{A+\psi(A)|A\in\mathfrak{h}\}\ltimes((\mathbb{C}^k\oplus\mathbb{R}^{n-l})\ltimes\mathbb{R})$,

where k and l are integers such that $0<k\leq l\leq n$, we have the decomposition $\mathbb{C}^n=\mathbb{C}^k\oplus\mathbb{C}^{l-k}\oplus\mathbb{C}^{n-l}$, $\mathfrak{h}\subset\mathfrak{u}(k)$ is a subalgebra with $\dim\mathfrak{z}(\mathfrak{h})\geq n+l-2k$ and $\psi:\mathfrak{h}\to\mathbb{C}^{l-k}\oplus i\mathbb{R}^{n-l}$ is a surjective linear map with $\psi|_{\mathfrak{h}'}=0$;

$\mathfrak{g}^{m,\mathfrak{h},\psi,k,l,r}= \{A+\psi(A)|A\in\mathfrak{h}\}\ltimes((\mathbb{C}^k\oplus\mathbb{R}^{m-l}\oplus\mathbb{R}^{r-m})\ltimes\mathbb{R})$,

where k, l, r and m are integers such that $0<k\leq l\leq m\leq r\leq n$ and $m<n$, we have the decomposition $\mathbb{C}^n=\mathbb{C}^k\oplus\mathbb{C}^{l-k}\oplus\mathbb{C}^{m-l}\oplus\mathbb{C}^{r-m}\oplus\mathbb{C}^{n-r}$, $\mathfrak{h}\subset\mathfrak{u}(k)\oplus\mathfrak{sod}(r-m)$ is a subalgebra with $\dim\mathfrak{z}(\mathfrak{h})\geq n+m+l-2k-r$ and $\psi:\mathfrak{h}\to\mathbb{C}^{l-k}\oplus\mathbb{R}^{n-r}\oplus i\mathbb{R}^{m-l}$ is a surjective linear map with $\psi|_{\mathfrak{h}'}=0$;

$\mathfrak{g}^{0,\mathfrak{h},\psi,k}= \{A+\psi(A)|A\in\mathfrak{h}\}\ltimes(\mathbb{R}^k\ltimes\mathbb{R})$,

where $0<k<n$, we have the decomposition $\mathbb{C}^n=\mathbb{C}^k\oplus\mathbb{C}^{n-k}$, $\mathfrak{h}\subset\mathfrak{sod}(k)$ is a subalgebra such that $\dim\mathfrak{z}(\mathfrak{h})\geq n-k$, $\psi:\mathfrak{h}\to\mathbb{R}^{n-k}$ is a surjective linear map with $\psi|_{\mathfrak{h}'}=0$;

$\mathfrak{g}^{0,\mathfrak{h},\zeta}= \{A+\zeta(A)|A\in\mathfrak{h}\}\ltimes\mathbb{R}^n$,

where $\mathfrak{h}\subset\mathfrak{sod}(n)$ is a subalgebra with $\mathfrak{z}(\mathfrak{h})\neq\{0\}$, $\zeta:\mathfrak{h}\to\mathbb{R}\subset\mathrm{LA}\,\mathcal{H}_n$ is a non-zero linear map with $\zeta|_{\mathfrak{z}(\mathfrak{h})}\neq 0$;

$\mathfrak{g}^{0,\mathfrak{h},\psi,k,\zeta}= \{A+\psi(A)+\zeta(A)|A\in\mathfrak{h}\}\ltimes\mathbb{R}^k$,

where $1\leq k<n$, we have the decomposition $\mathbb{C}^n=\mathbb{C}^k\oplus\mathbb{C}^{n-k}$, $\mathfrak{h}\subset\mathfrak{sod}(k)$ is a subalgebra with $\dim\mathfrak{z}(\mathfrak{h})\geq n-k$, $\psi:\mathfrak{h}\to\mathbb{R}^{n-k}$ is a surjective linear map with $\psi|_{\mathfrak{h}'}=0$, $\zeta:\mathfrak{h}\to\mathbb{R}\subset\mathrm{LA}\,\mathcal{H}_n$ is a non-zero linear map with $\zeta|_{\mathfrak{h}'}=0$.

Note that the last two types of weakly-irreducible subalgebras $\mathfrak{g}\subset\mathfrak{su}(1,n+1)_l\subset\mathfrak{so}(2,2n+2)$ were not considered by A. Ikemakhen in [15].

For each $\mathfrak{f}\subset\mathrm{LA}(\mathrm{Sim}\,\mathcal{H}_n)$ as above and for each $\mathfrak{g}\subset\mathfrak{su}(1,n+1)_l$ with $\pi(\Gamma(\mathfrak{g}))=\mathfrak{f}$ we consider the Lie algebras $\mathfrak{g}^J=\mathfrak{g}\oplus\mathbb{R}J$ and $\mathfrak{g}^\xi=\{x+\xi(x)|x\in\mathfrak{g}\}$, where $\xi:\mathfrak{g}\to\mathbb{R}$ is a non-zero linear map. As we claimed above, any weakly-irreducible subalgebra of $\mathfrak{u}(1,n+1)_l\subset\mathfrak{so}(2,2n+2)$ is of the form $\mathfrak{g}$, $\mathfrak{g}^J$ or $\mathfrak{g}^\xi$. These subalgebras are candidates for the weakly-irreducible subalgebras of $\mathfrak{u}(1,n+1)_l\subset\mathfrak{so}(2,2n+2)$. We associate with each of these subalgebras an integer $0\leq m\leq n$. If $m>0$, then the subalgebras of the form $\mathfrak{g}$, $\mathfrak{g}^J$

and $\mathfrak{g}^{\xi} \subset \mathfrak{u}(1,n+1)_l$ are weakly-irreducible. We have inclusions $\mathfrak{u}(m) \subset \mathfrak{u}(n) \subset \mathfrak{u}(1,n+1)_l$ and projection maps $\mathrm{pr}_{\mathfrak{u}(m)} : \mathfrak{u}(1,n+1)_l \to \mathfrak{u}(m)$, $\mathrm{pr}_{\mathfrak{u}(n)} : \mathfrak{u}(1,n+1)_l \to \mathfrak{u}(n)$.

In **Section 3** we classify weakly-irreducible Berger subalgebras of $\mathfrak{u}(1,n+1)_l$. These subalgebras are candidates for the holonomy algebras. Then we show that all these Berger algebras are holonomy algebras.

More precisely, for any subalgebra $\mathfrak{g} \subset \mathfrak{u}(1,n+1)_l$ consider the space $\mathcal{R}(\mathfrak{g})$ of curvature tensors of type $\mathfrak{g}$,

$$\mathcal{R}(\mathfrak{g}) = \left\{ R \in \mathrm{Hom}(\mathbb{R}^{2,2n+2} \wedge \mathbb{R}^{2,2n+2}, \mathfrak{g}) \;\middle|\; \begin{array}{c} R(u\wedge v)w + R(v\wedge w)u + R(w\wedge u)v = 0 \\ \text{for all } u,v,w \in \mathbb{R}^{2,2n+2} \end{array} \right\}.$$

Denote by $L(\mathcal{R}(\mathfrak{g}))$ the vector subspace of $\mathfrak{g}$ spanned by $R(u\wedge v)$ for all $R \in \mathcal{R}(\mathfrak{g})$, $u, v \in \mathbb{R}^{2,2n+2}$,

$$L(\mathcal{R}(\mathfrak{g})) = \mathrm{span}\{R(u\wedge v) | R \in \mathcal{R}(\mathfrak{g}),\ u,v \in \mathbb{R}^{2,2n+2}\}.$$

A subalgebra $\mathfrak{g} \subset \mathfrak{u}(1,n+1)_l$ is called *a Berger algebra* if $L(\mathcal{R}(\mathfrak{g})) = \mathfrak{g}$. From the Ambrose-Singer theorem [4] it follows that *if* $\mathfrak{g} \subset \mathfrak{u}(1,n+1)_l$ *is the holonomy algebra of a pseudo-Kählerian manifold, then* $\mathfrak{g}$ *is a Berger algebra* (here we identify the tangent space to the manifold at some point with $\mathbb{R}^{2,2n+2}$).

First we consider all subalgebras of $\mathfrak{u}(1,1)_l$ and show which of these subalgebras are weakly-irreducible Berger subalgebras.

Then we consider the case $n \geq 1$. For any integer $0 \leq m \leq n$ and subalgebra $\mathfrak{u} \subset \mathfrak{u}(m) \oplus \mathfrak{sod}(m+1,...,n)$ we consider a subalgebra $\mathfrak{g}^{m,\mathfrak{u}} \subset \mathfrak{u}(1,n+1)_l$ and describe the space $\mathcal{R}(\mathfrak{g}^{m,\mathfrak{u}})$. The Lie algebras of the form $\mathfrak{g}^{m,\mathfrak{u}}$ contain all candidates for the weakly-irreducible subalgebras of $\mathfrak{u}(1,n+1)_l$. For any subalgebra $\mathfrak{g} \subset \mathfrak{g}^{m,\mathfrak{u}}$ the space $\mathcal{R}(\mathfrak{g})$ can be found from the following condition

$$R \in \mathcal{R}(\mathfrak{g}) \text{ if and only if } R \in \mathcal{R}(\mathfrak{g}^{m,\mathfrak{u}}) \text{ and } R(\mathbb{R}^{2,2n+2} \wedge \mathbb{R}^{2,2n+2}) \subset \mathfrak{g}.$$

Using this, we easily find all weakly-irreducible not irreducible Berger subalgebras of $\mathfrak{u}(1,n+1)_l$.

As the last step of the classification, we construct metrics on $\mathbb{R}^{2n+4}$ that realize all Berger algebras obtained above as holonomy algebras. The coefficients of the metrics are polynomial functions, hence the corresponding Levi-Civita connections are analytic and in each

case the holonomy algebra at the point $0 \in \mathbb{R}^{2n+4}$ is generated by the operators

$$R(X,Y)_0, \nabla_{Z_1} R(X,Y)_0, \nabla_{Z_2}\nabla_{Z_1} R(X,Y)_0, \dots$$

where X, Y, Z_1, Z_2,... are vectors at the point 0. We explicitly compute for each metric the components of the curvature tensor and its derivatives. Then using the induction, we find the holonomy algebra for each of the metrics.

Thus we obtain the classification of weakly-irreducible not irreducible holonomy algebras contained in $\mathfrak{u}(1,n+1)$. The result can be stated as follows.

Let $n=0$. The Lie algebra $\mathfrak{u}(1,1)_l$ is a 3-dimensional nilpotent real Lie algebra, we have $\mathfrak{u}(1,1)_l = \mathbb{C} \ltimes \mathbb{R}$ and $[(a+ib,0),(0,c)] = (0,2ac)$, a, b, $c \in \mathbb{R}$. There are three weakly-irreducible holonomy algebras contained in $\mathfrak{u}(1,1)_l$:

$$\mathfrak{hol}^1_{n=0} = \mathfrak{u}(1,1)_l, \quad \mathfrak{hol}^{\gamma_1,\gamma_2}_{n=0} = \mathbb{R}(\gamma_1 + i\gamma_2) \ltimes \mathbb{R}\,(\gamma_1,\gamma_2 \in \mathbb{R}), \quad \mathfrak{hol}^2_{n=0} = \mathbb{C}.$$

Let $n>0$. For the Lie algebra $\mathfrak{u}(1,n+1)_l$ we have the Iwasawa decomposition

$$\mathfrak{u}(1,n+1)_l = (\mathbb{C} \oplus \mathfrak{u}(n)) \ltimes \mathrm{LA}\,\mathcal{H}_n.$$

Consider the following type of weakly-irreducible holonomy algebras contained in $\mathfrak{u}(1,n+1)_l \subset \mathfrak{so}(2,2n+2)$:

$$\mathfrak{hol}^{m,\mathfrak{u},\mathcal{A}^1,\tilde{\mathcal{A}}^2} = (\mathbb{R} \oplus \mathbb{R}(i+J_{n-m}) \oplus \mathfrak{u}) \ltimes ((\mathbb{C}^m \oplus \mathbb{R}^{n-m}) \ltimes \mathbb{R}),$$

where $0 \le m \le n$ is an integer, we have the decompositions $\mathbb{C}^n = \mathbb{C}^m \oplus \mathbb{C}^{n-m}$, $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$, $\mathfrak{u} \subset \mathfrak{u}(m)$ is a subalgebra, $\mathbb{R}^{n-m} \subset \mathbb{C}^{n-m}$ is a real form and $J_{n-m} \subset \mathfrak{u}(n-m) \subset \mathfrak{u}(n) \subset \mathfrak{u}(1,n+1)_l$ is the complex structure on $\mathbb{C}^{n-m}$.

The other types of weakly-irreducible holonomy algebras contained in $\mathfrak{u}(1,n+1)_l \subset \mathfrak{so}(2,2n+2)$ can be obtained from this type using some twisting. Let $\varphi, \phi : \mathfrak{u} \to \mathbb{R}$ be linear maps with $\varphi|_{\mathfrak{u}'} = \phi|_{\mathfrak{u}'} = 0$. The other types have the following forms:

$\mathfrak{hol}^{m,\mathfrak{u},\mathcal{A}^1,\phi} = (\mathbb{R} \oplus \{\phi(A)(i+J_{n-m}) + A | A \in \mathfrak{u}\}) \ltimes ((\mathbb{C}^m \oplus \mathbb{R}^{n-m}) \ltimes \mathbb{R})$,

$\mathfrak{hol}^{m,\mathfrak{u},\varphi,\phi} = \{\varphi(A) + \phi(A)(i+J_{n-m}) + A | A \in \mathfrak{u}\} \ltimes ((\mathbb{C}^m \oplus \mathbb{R}^{n-m}) \ltimes \mathbb{R})$,

$\mathfrak{hol}^{m,\mathfrak{u},\varphi,\tilde{\mathcal{A}}^2} = (\mathbb{R}(i+J_{n-m}) \oplus \{\varphi(A) + A | A \in \mathfrak{u}\}) \ltimes ((\mathbb{C}^m \oplus \mathbb{R}^{n-m}) \ltimes \mathbb{R})$,

$\mathfrak{hol}^{m,\mathfrak{u},\lambda} = (\mathbb{R}(1+\lambda(i+J_{n-m})) \oplus \mathfrak{u}) \ltimes ((\mathbb{C}^m \oplus \mathbb{R}^{n-m}) \ltimes \mathbb{R})$, where $\lambda \in \mathbb{R}$,

$\mathfrak{hol}^{n,\mathfrak{u},\psi,k,l}= \{A+\psi(A)|A\in\mathfrak{u}\}\ltimes((\mathbb{C}^k\oplus\mathbb{R}^{n-l})\ltimes\mathbb{R})$,

where k and l are integers such that $0<k\leq l\leq n$, we have the decomposition $\mathbb{C}^n=\mathbb{C}^k\oplus\mathbb{C}^{l-k}\oplus\mathbb{C}^{n-l}$, $\mathfrak{u}\subset\mathfrak{u}(k)$ is a subalgebra with $\dim\mathfrak{z}(\mathfrak{u})\geq n+l-2k$ and $\psi:\mathfrak{u}\to\mathbb{C}^{l-k}\oplus i\mathbb{R}^{n-l}$ is a surjective linear map with $\psi|_{\mathfrak{u}'}=0$,

$\mathfrak{hol}^{m,\mathfrak{u},\psi,k,l,r}= \{A+\psi(A)|A\in\mathfrak{u}\}\ltimes((\mathbb{C}^k\oplus\mathbb{R}^{m-l}\oplus\mathbb{R}^{r-m})\ltimes\mathbb{R})$,

where k, l, r and m are integers such that $0<k\leq l\leq m\leq r\leq n$ and $m<n$, we have the decomposition $\mathbb{C}^n=\mathbb{C}^k\oplus\mathbb{C}^{l-k}\oplus\mathbb{C}^{m-l}\oplus\mathbb{C}^{r-m}\oplus\mathbb{C}^{n-r}$, $\mathfrak{u}\subset\mathfrak{u}(k)$ is a subalgebra with $\dim\mathfrak{z}(\mathfrak{u})\geq n+m+l-2k-r$ and $\psi:\mathfrak{u}\to\mathbb{C}^{l-k}\oplus i\mathbb{R}^{m-l}\oplus\mathbb{R}^{n-r}$ is a surjective linear map with $\psi|_{\mathfrak{u}'}=0$.

As a corollary, we get the classification of weakly-irreducible not irreducible holonomy algebras contained in $\mathfrak{su}(1,n+1)$ (i.e. of the holonomy algebras of special pseudo-Kählerian manifolds). For $n=0$ these algebras are exhausted by $\{(0,c)|c\in\mathbb{R}\}$ and $\mathfrak{su}(1,1)_l$. For $n>0$ these algebras are the following:

$\mathfrak{hol}^{m,\mathfrak{u},\mathcal{A}^1,\phi}$, $\mathfrak{hol}^{m,\mathfrak{u},\varphi,\phi}$ with $\phi(A)=-\frac{1}{n-m+2}\operatorname{tr}_{\mathbb{C}}A$;

$\mathfrak{hol}^{n,\mathfrak{u},\psi,k,l}$, $\mathfrak{hol}^{m,\mathfrak{u},\psi,k,l,r}$ with $\mathfrak{u}\subset\mathfrak{su}(k)$.

The above result together with the Wu theorem and the classification of irreducible holonomy algebras of M. Berger gives us the classification of holonomy algebras (or equivalently, of connected holonomy groups) for pseudo-Kählerian manifolds of signature $(2,2n+2)$.

Remark that we do not have any additional condition on the $\mathfrak{u}(m)$-projection of a holonomy algebra, while in the Lorentzian case an analogous subalgebra $\mathfrak{h}\subset\mathfrak{so}(n)$ associated to a holonomy algebra must be the holonomy algebra of a Riemannian manifold. This shows the principal difference between our case and the case of Lorentzian manifolds.

2 Weakly-irreducible not irreducible subalgebras of $\mathfrak{su}(1,n+1)$

In this chapter we classify weakly-irreducible not irreducible subalgebras of $\mathfrak{su}(1,n+1)$. The result is stated in Section 2.1. In the other sections we give preliminaries and the proof of the main theorem.

2.1 Classification of weakly-irreducible not irreducible subalgebras of $\mathfrak{su}(1,n+1)$

Let $\mathbb{R}^{2,2n+2}$ be a $2n+4$-dimensional real vector space endowed with a complex structure $J\in\operatorname{Aut}\mathbb{R}^{2,2n+2}$, $J^2=-\operatorname{id}$ and with a J-invariant metric η of signature $(2,2n+2)$, i.e. $\eta(Jx,Jy)=\eta(x,y)$ for all $x,y\in\mathbb{R}^{2,2n+2}$. We fix a basis p_1, p_2, e_1,...,e_n, f_1,...,f_n, q_1, q_2 of the vector space $\mathbb{R}^{2,2n+2}$ such that the Gram matrix of the metric η and the complex structure J have the forms

$$\begin{pmatrix} 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&E_{2n}&0&0\\ 1&0&0&0&0\\ 0&1&0&0&0 \end{pmatrix} \text{ and } \begin{pmatrix} 0&-1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&-E_n&0&0\\ 0&0&E_n&0&0&0\\ 0&0&0&0&0&-1\\ 0&0&0&0&1&0 \end{pmatrix}, \text{ respectively.}$$

We denote by $\mathfrak{u}(1,n+1)_{<p_1,p_2>}$ the subalgebra of $\mathfrak{u}(1,n+1)$ that preserves the J-invariant 2-dimensional isotropic subspace $\mathbb{R}p_1\oplus\mathbb{R}p_2\subset\mathbb{R}^{2,2n+2}$. The Lie algebra $\mathfrak{u}(1,n+1)_{<p_1,p_2>}$ can be identified with the following matrix algebra

$$\mathfrak{u}(1,n+1)_{<p_1,p_2>}=\left\{\left.\begin{pmatrix} a_1&-a_2&-z_1^t&-z_2^t&0&-c\\ a_2&a_1&z_2^t&-z_1^t&c&0\\ 0&0&B&-C&z_1&-z_2\\ 0&0&C&B&z_2&z_1\\ 0&0&0&0&-a_1&-a_2\\ 0&0&0&0&a_2&-a_1 \end{pmatrix}\right| \begin{array}{c} a_1,a_2,c\in\mathbb{R},\\ z_1,z_2\in\mathbb{R}^n,\\ \left(\begin{smallmatrix} B&-C\\ C&B\end{smallmatrix}\right)\in\mathfrak{u}(n)\end{array}\right\}.$$

Recall that

$$\mathfrak{u}(n)=\left\{\left(\begin{smallmatrix} B&-C\\ C&B\end{smallmatrix}\right)\middle|\, B\in\mathfrak{so}(n), C\in\mathfrak{gl}(n), C^t=C\right\}$$

and

$$\mathfrak{su}(n)=\left\{\left(\begin{smallmatrix} B&-C\\ C&B\end{smallmatrix}\right)\in\mathfrak{u}(n)\middle|\,\operatorname{tr}C=0\right\}.$$

We identify the above element of $\mathfrak{u}(1,n+1)_{<p_1,p_2>}$ with the 7-tuple (a_1,a_2,B,C,z_1,z_2,c). Define the following vector subspaces of $\mathfrak{u}(1,n+1)_{<p_1,p_2>}$:

$$\begin{aligned} \mathcal{A}^1 &= \{(a_1,0,0,0,0,0,0)|a_1\in\mathbb{R}\}, & \mathcal{A}^2 &= \{(0,a_2,0,0,0,0,0)|a_2\in\mathbb{R}\},\\ \mathcal{N}^1 &= \{(0,0,0,0,z_1,0,0)|z_1\in\mathbb{R}^n\}, & \mathcal{N}^2 &= \{(0,0,0,0,0,z_2,0)|z_2\in\mathbb{R}^n\} \end{aligned}$$

and

$$\mathcal{C}=\{(0,0,0,0,0,0,c)|c\in\mathbb{R}\}.$$

We consider $\mathfrak{u}(n)$ as a subalgebra of $\mathfrak{u}(1,n+1)_{<p_1,p_2>}$.

We see that $\mathcal{C}$ is a commutative ideal, which commutes with $\mathcal{A}^2$, $\mathcal{N}^1$, $\mathcal{N}^2$ and $\mathfrak{u}(n)$, and $\mathcal{A}^1 \oplus \mathcal{A}^2$ is a commutative subalgebras, which commutes with $\mathfrak{u}(n)$.

Furthermore, for $a_1, a_2, c \in \mathbb{R}$, $z_1, z_2, w_1, w_2 \in \mathbb{R}^n$ and $\left(\begin{smallmatrix} B & -C \\ C & B \end{smallmatrix}\right) \in \mathfrak{u}(n)$ we obtain

$$\begin{aligned}
[(a_1,0,0,0,0,0,0),(0,0,0,0,z_1,z_2,c)] &= (0,0,0,0,a_1z_1,a_1z_2,2a_1c), \\
[(0,a_2,0,0,0,0,0),(0,0,0,0,z_1,z_2,0)] &= (0,0,0,0,a_2z_2,-a_2z_1,0), \\
[(0,0,B,C,0,0,0),(0,0,0,0,z_1,z_2,0)] &= (0,0,0,0,Bz_1-Cz_2,Cz_1+Bz_2,0), \\
[(0,0,0,0,z_1,z_2,0),(0,0,0,0,w_1,w_2,0)] &= (0,0,0,0,0,0,2(-z_1w_2^t+z_2w_1^t)).
\end{aligned}$$

Hence we obtain the decomposition[1]

$$\mathfrak{u}(1,n+1)_{<p_1,p_2>} = (\mathcal{A}^1 \oplus \mathcal{A}^2 \oplus \mathfrak{u}(n)) \ltimes (\mathcal{N}^1 + \mathcal{N}^2 + \mathcal{C}).$$

Denote by $\mathfrak{su}(1,n+1)_{<p_1,p_2>}$ the subalgebra of $\mathfrak{su}(1,n+1)$ that preserves the subspace $\mathbb{R}p_1 \oplus \mathbb{R}p_2 \subset \mathbb{R}^{2,2n+2}$. Then

$$\mathfrak{su}(1,n+1)_{<p_1,p_2>} = \{(a_1,a_2,B,C,z_1,z_2,c) \in \mathfrak{u}(1,n+1)_{<p_1,p_2>} | 2a_2 + \mathrm{tr}_{\mathbb{R}}\, C = 0\}$$

and

$$\mathfrak{u}(1,n+1)_{<p_1,p_2>} = \mathfrak{su}(1,n+1)_{<p_1,p_2>} \oplus \mathbb{R}J.$$

Therefore we obtain the decomposition

$$\mathfrak{su}(1,n+1)_{<p_1,p_2>} = (\mathcal{A}^1 \oplus \mathfrak{su}(n) \oplus \mathbb{R}I_0) \ltimes (\mathcal{N}^1 + \mathcal{N}^2 + \mathcal{C}),$$

where

$$I_0 = \begin{pmatrix}
0 & \frac{n}{n+2} & 0 & 0 & 0 & 0 \\
-\frac{n}{n+2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{2}{n+2}E_n & 0 & 0 \\
0 & 0 & \frac{2}{n+2}E_n & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{n}{n+2} \\
0 & 0 & 0 & 0 & -\frac{n}{n+2} & 0
\end{pmatrix}.$$

Note that

$$\mathfrak{u}(1,1)_{<p_1,p_2>} = \left\{ \begin{pmatrix} a_1 & -a_2 & 0 & -c \\ a_2 & a_1 & c & 0 \\ 0 & 0 & -a_1 & -a_2 \\ 0 & 0 & a_2 & -a_1 \end{pmatrix} \middle| \, a_1, a_2, c \in \mathbb{R} \right\} = (\mathcal{A}^1 \oplus \mathcal{A}^2) \ltimes \mathcal{C}$$

and $\mathfrak{su}(1,1)_{<p_1,p_2>} = \mathcal{A}^1 \ltimes \mathcal{C}$.

[1]We denote by $\rtimes$ and $\ltimes$ the semi-direct product and semi-direct sum for Lie groups and Lie algebras, respectively. If a Lie algebra is decomposed into a direct sum of vector subspaces, then we use $+$.

If a weakly-irreducible subalgebra $\mathfrak{g} \subset \mathfrak{u}(1, n+1)$ preserves a degenerate proper subspace $W \subset \mathbb{R}^{2,2n+2}$, then $\mathfrak{g}$ preserves the J-invariant 2-dimensional isotropic subspace $W_1 \subset \mathbb{R}^{2,2n+2}$, where $W_1 = (W \cap JW) \cap (W \cap JW)^\perp$ if $W \cap JW \neq \{0\}$ and $W_1 = (W \oplus JW) \cap (W \oplus JW)^\perp$ if $W \cap JW = \{0\}$. Therefore $\mathfrak{g}$ is conjugated to a weakly-irreducible subalgebra of $\mathfrak{u}(1, n+1)_{<\mathbb{R}p_1, \mathbb{R}p_2>}$.

Let $E = \mathrm{span}\{e_1, ..., e_n, f_1, ..., f_n\}$, $E^1 = \mathrm{span}\{e_1, ..., e_n\}$ and $E^2 = \mathrm{span}\{f_1, ..., f_n\}$. For any integers k and l with $1 \leq k \leq l \leq n$ we consider the following subspaces:

$$E^1_{k,...,l} = \mathrm{span}\{e_k, ..., e_l\} \subset E^1, E^2_{k,...,l} = \mathrm{span}\{f_k, ..., f_l\} \subset E^2, E_{k,...,l} = E^1_{k,...,l} \oplus E^2_{k,...,l} \subset E,$$

$$\mathcal{N}^1_{k,...,l} = \{(0,0,0,0,z_1,0,0) | z_1 \in E^1_{k,...,l}\} \subset \mathcal{N}^1$$

and

$$\mathcal{N}^2_{k,...,l} = \{(0,0,0,0,0,z_2,0) | z_2 \in E^2_{k,...,l}\} \subset \mathcal{N}^2.$$

Clearly, $E^1 = E^1_{1,...,n}$, $E^2 = E^2_{1,...,n}$, $E = E_{1,...,n}$, $\mathcal{N}^1 = \mathcal{N}^1_{1,...,n}$ and $\mathcal{N}^2 = \mathcal{N}^2_{1,...,n}$.

We denote by $\mathfrak{u}(e_k, ..., e_l)$ the subalgebra of $\mathfrak{u}(n)$ that preserves the vector subspace $E_{k,...,l} \subset E$ and annihilates the orthogonal complement to this subspace. We denote $\mathfrak{u}(1, ..., l)$ just by $\mathfrak{u}(l)$. Furthermore, let $J_{k,...,l}$ be the element of $\mathfrak{u}(1, n+1)_{<\mathbb{R}p_1, \mathbb{R}p_2>}$ defined by $J_{k,...,l}|_{E_{k,...,l}} = J|_{E_{k,...,l}}$ and $J_{k,...,l}|_{E^\perp_{k,...,l}} = 0$. We denote $J_{1,...,l}$ just by J_l. Consider the following Lie algebra

$$\mathfrak{sod}(k, ..., l) = \left\{ \left. \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right| B \in \mathfrak{so}(l-k+1) \right\} \subset \mathfrak{su}(k, ..., l),$$

where the matrices of the operators are written with respect to the decomposition

$$E = E^1_{1,...,k-1} \oplus E^1_{k,...,l} \oplus E^1_{l+1,...,n} \oplus E^2_{1,...,k-1} \oplus E^2_{k,...,l} \oplus E^2_{l+1,...,n}.$$

The subalgebra $\mathfrak{sod}(k, ..., l) \subset \mathfrak{u}(n)$ annihilates the orthogonal complement to $E^1_{k,...,l} \oplus E^2_{k,...,l}$ and acts diagonally on $E^1_{k,...,l} \oplus E^2_{k,...,l}$.

For any $0 \leq m \leq n$ define the following vector space

$$\tilde{\mathcal{A}}^2 = \{(0, a_2, 0, 0, 0, 0, 0) + a_2 J_{m+1,...,n} | a_2 \in \mathbb{R}\}.$$

Clearly, if $m = n$, then $\tilde{\mathcal{A}}^2 = \mathcal{A}^2$; if $m = 0$, then $\tilde{\mathcal{A}}^2 = \mathbb{R}J$.

Let $\mathfrak{h} \subset \mathfrak{u}(n)$ be a subalgebra. Recall that $\mathfrak{h}$ is a compact Lie algebra and we have the decomposition $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{z}(\mathfrak{h})$, where $\mathfrak{h}'$ is the commutant of $\mathfrak{h}$ and $\mathfrak{z}(\mathfrak{h})$ is the center of $\mathfrak{h}$, see for example [21].

Theorem 2.1. 1) *A subalgebra $\mathfrak{g} \subset \mathfrak{su}(1,1) \subset \mathfrak{so}(2,2)$ is weakly-irreducible and not irreducible if and only if $\mathfrak{g}$ is conjugated to the subalgebra $\mathcal{C} \subset \mathfrak{su}(1,1)_{<p_1,p_2>}$ or to $\mathfrak{su}(1,1)_{<p_1,p_2>}$.*

2) *Let $n \geq 1$. Then a subalgebra $\mathfrak{g} \subset \mathfrak{su}(1,n+1) \subset \mathfrak{so}(2,2n+2)$ is weakly-irreducible and not irreducible if and only if $\mathfrak{g}$ is conjugated to one of the following subalgebras of $\mathfrak{su}(1,n+1)_{<p_1,p_2>}$:*

$\mathfrak{g}^{m,\mathfrak{h},\mathcal{A}^1} = (\mathcal{A}^1 \oplus \{(0,-\frac{1}{2}\operatorname{tr} C, B, C, 0,0,0) | \left(\begin{smallmatrix} B & -C \\ C & B \end{smallmatrix}\right) \in \mathfrak{h}\}) \ltimes (\mathcal{N}^1 + \mathcal{N}^2_{1,\dots,m} + \mathcal{C})$,
where $0 \leq m \leq n$ and $\mathfrak{h} \subset \mathfrak{su}(m) \oplus (J_m - \frac{m}{n+2}J_n) \oplus \mathfrak{sod}(m+1,\dots,n)$ is a subalgebra;

$\mathfrak{g}^{m,\mathfrak{h},\varphi} = \{(\varphi(B,C), -\frac{1}{2}\operatorname{tr} C, B, C, 0,0,0) | \left(\begin{smallmatrix} B & -C \\ C & B \end{smallmatrix}\right) \in \mathfrak{h}\} \ltimes (\mathcal{N}^1 + \mathcal{N}^2_{1,\dots,m} + \mathcal{C})$,
where $0 \leq m \leq n$, $\mathfrak{h} \subset \mathfrak{su}(m) \oplus (J_m - \frac{m}{n+2}J_n) \oplus \mathfrak{sod}(m+1,\dots,n)$ is a subalgebra and $\varphi : \mathfrak{h} \to \mathbb{R}$ is a linear map with $\varphi|_{\mathfrak{h}'} = 0$;

$$\begin{aligned}\mathfrak{g}^{n,\mathfrak{h},\psi,k,l} = \ & \{(0,-\tfrac{1}{2}\operatorname{tr} C, B, C, \psi_1(B,C), \psi_2(B,C)+\psi_3(B,C), 0) | \left(\begin{smallmatrix} B & -C \\ C & B \end{smallmatrix}\right) \in \mathfrak{h}\} \\ & \ltimes (\mathcal{N}^1_{1,\dots,k} + \mathcal{N}^2_{1,\dots,k} + \mathcal{N}^1_{l+1,\dots,n} + \mathcal{C}),\end{aligned}$$
where k and l are integers such that $0 < k \leq l \leq n$, $\mathfrak{h} \subset \mathfrak{su}(k) \oplus \mathbb{R}(J_k - \frac{k}{n+2}J_n)$ is a subalgebra with $\dim \mathfrak{z}(\mathfrak{h}) \geq n + l - 2k$, $\psi : \mathfrak{h} \to E^1_{k+1,\dots,l} + E^2_{k+1,\dots,l} + E^2_{l+1,\dots,n}$ is a surjective linear map with $\psi|_{\mathfrak{h}'} = 0$, $\psi_1 = \operatorname{pr}_{E^1_{k+1,\dots,l}} \circ \psi$, $\psi_2 = \operatorname{pr}_{E^2_{k+1,\dots,l}} \circ \psi$ and $\psi_3 = \operatorname{pr}_{E^2_{l+1,\dots,n}} \circ \psi$;

$$\begin{aligned}\mathfrak{g}^{m,\mathfrak{h},\psi,k,l,r} = \ & \{(0,-\tfrac{1}{2}\operatorname{tr} C, B, C, \psi_1(B,C)+\psi_4(B,C), \psi_2(B,C)+\psi_3(B,C), 0) | \left(\begin{smallmatrix} B & -C \\ C & B \end{smallmatrix}\right) \in \mathfrak{h}\} \\ & \ltimes (\mathcal{N}^1_{1,\dots,k} + \mathcal{N}^2_{1,\dots,k} + \mathcal{N}^1_{l+1,\dots,m} + \mathcal{N}^1_{m+1,\dots,r} + \mathcal{C}),\end{aligned}$$

where k, l, r and m are integers such that $0 < k \leq l \leq m \leq r \leq n$ and $m < n$, $\mathfrak{h} \subset \mathfrak{su}(k) \oplus \mathbb{R}(J_k - \frac{k}{n+2}J_n) \oplus \mathfrak{sod}(m+1,\dots,r)$ is a subalgebra with $\dim \mathfrak{z}(\mathfrak{h}) \geq n + m + l - 2k - r$, $\psi : \mathfrak{h} \to E^1_{k+1,\dots,l} + E^2_{k+1,\dots,l} + E^2_{l+1,\dots,m} + E^1_{r+1,\dots,n}$ is a surjective linear map with $\psi|_{\mathfrak{h}'} = 0$, $\psi_1 = \operatorname{pr}_{E^1_{k+1,\dots,l}} \circ \psi$, $\psi_2 = \operatorname{pr}_{E^2_{k+1,\dots,l}} \circ \psi$, $\psi_3 = \operatorname{pr}_{E^2_{l+1,\dots,m}} \circ \psi$ and $\psi_4 = \operatorname{pr}_{E^1_{r+1,\dots,n}} \circ \psi$;

$\mathfrak{g}^{0,\mathfrak{h},\psi,k} = \{(0,0,B,0,\psi(B),0,0) | \left(\begin{smallmatrix} B & 0 \\ 0 & B \end{smallmatrix}\right) \in \mathfrak{h}\} \ltimes (\mathcal{N}^1_{1,\dots,k} + \mathcal{C})$,
where $0 < k < n$, $\mathfrak{h} \subset \mathfrak{sod}(1,\dots,k)$ is a subalgebra such that $\dim \mathfrak{z}(\mathfrak{h}) \geq n - k$, $\psi : \mathfrak{h} \to E^1_{k+1,\dots,n}$ is a surjective linear map with $\psi|_{\mathfrak{h}'} = 0$;

$\mathfrak{g}^{0,\mathfrak{h},\zeta}= \{(0,0,B,0,0,0,\zeta(B))|\left(\begin{smallmatrix} B & 0\\ 0 & B\end{smallmatrix}\right) \in \mathfrak{h}\} \ltimes \mathcal{N}^1$,

where $\mathfrak{h} \subset \mathfrak{sod}(1,...,n)$ *is a subalgebra with* $\mathfrak{z}(\mathfrak{h}) \neq \{0\}$ *and* $\zeta : \mathfrak{h} \to \mathbb{R}$ *is a non-zero linear map with* $\zeta|_{\mathfrak{z}(\mathfrak{h})} \neq 0$*;*

$\mathfrak{g}^{0,\mathfrak{h},\psi,k,\zeta}= \{(0,0,B,0,\psi(B),0,\zeta(B))|\left(\begin{smallmatrix} B & 0\\ 0 & B\end{smallmatrix}\right) \in \mathfrak{h},\} \ltimes \mathcal{N}^1_{1,\dots,k}$,

where $1 \leq k < n$, $\mathfrak{h} \subset \mathfrak{sod}(1,...,k)$ *is a subalgebra with* $\dim \mathfrak{z}(\mathfrak{h}) \geq n-k$, $\psi : \mathfrak{h} \to E^1_{k+1,\dots,n}$ *is a surjective linear map with* $\psi|_{\mathfrak{h}'} = 0$ *and* $\zeta : \mathfrak{h} \to \mathbb{R}$ *is a non-zero linear map with* $\zeta|_{\mathfrak{h}'} = 0$*.*

Remark. In [15] A. Ikemakhen classified weakly-irreducible subalgebras of $\mathfrak{so}(2,N+2)_{<p_1,p_2>}$ that contain the ideal $\mathcal{C}$, i.e. the last two types of Lie algebras from the above theorem were not considered in [15].

2.2 Transitive similarity transformation groups of Euclidian spaces

Let (E,η) be an Euclidean space, $\|x\|^2 = \eta(x,x)$. A map $f : E \to E$ is called a *similarity transformation* of E if there exists a $\lambda > 0$ such that $\|f(x_1) - f(x_2)\| = \lambda\|x_1 - x_2\|$ for all $x_1, x_2 \in E$. If $\lambda = 1$, then f is called an *isometry*. Denote by $\operatorname{Sim} E$ and $\operatorname{Isom} E$ the groups of all similarity transformations and isometries of E, respectively. A subgroup $G \subset \operatorname{Sim} E$ such that $G \not\subset \operatorname{Isom} E$ is called *essential*. A subgroup $G \subset \operatorname{Sim} E$ is called *irreducible* if it does not preserve any proper affine subspace of E.

The following theorem is due to D.V. Alekseevsky (see [2] or [3]).

Theorem 2.2. *A subgroup* $G \subset \operatorname{Sim} E$ *acts irreducibly on* E *if and only if it acts transitively.*

We denote by E the group of all translations in E and by $A^1 = \mathbb{R}^+$ the identity component of the group of all dilations of E about the origin. For the connected identity component of the Lie group $\operatorname{Isom} E$ we have the decomposition

$$\operatorname{Isom}^0 E = SO(E) \leftthreetimes E,$$

where E is a normal subgroup of $\operatorname{Isom}^0 E$. For $\operatorname{Sim}^0 E$ we obtain

$$\operatorname{Sim}^0 E = A^1 \leftthreetimes \operatorname{Isom}^0 E = (A^1 \times SO(E)) \leftthreetimes E,$$

where E is a normal subgroup of $\operatorname{Sim}^0 E$ and A^1 commutes with $SO(E)$.

In [12] we deduced from results of [2] and [3] the following theorem.

Theorem 2.3. *Let $G \subset \operatorname{Sim} E$ be a transitively acting connected subgroup. Then G belongs to one of the following types*

type 1. $G = (A \times H) \leftthreetimes E$, *where $H \subset SO(E)$ is a connected Lie subgroup;*

type 2. $G = H \leftthreetimes E$;

type 3. $G = (A^\Phi \times H) \leftthreetimes E$, *where $\Phi : A \to SO(E)$ is a non-trivial homomorphism and*

$$A^\Phi = \{a \cdot \Phi(a) | a \in A\} \subset A \times SO(E)$$

is a group of screw dilations of E that commutes with H;

type 4. $G = (H \times U^\Psi) \leftthreetimes W$, *where $H \subset SO(W)$ is a connected Lie subgroup, $E = W \oplus U$ is an orthogonal decomposition, $\Psi : U \to SO(W)$ is a homomorphism with $\ker \mathrm{d}\,\Psi = \{0\}$, and*

$$U^\Psi = \{\Psi(u) \cdot u | u \in U\} \subset SO(W) \times U$$

is a group of screw isometries of E that commutes with H.

Remark. Consider the Minkowski space $\mathbb{R}^{1,n+1}$. Let $l \subset \mathbb{R}^{1,n+1}$ be an isotropic line and $E \subset \mathbb{R}^{1,n+1}$ be an Euclidean subspace orthogonal to l. In [12] was constructed an isomorphism $SO(1, n+1)_l \simeq \operatorname{Sim} E$, where $SO(1, n+1)_l$ is the connected subgroup of $SO(1, n+1)$ that preserves the line l. It was proved that a subgroup $G \subset SO(1, n+1)_l$ is weakly-irreducible if and only if the corresponding subgroup of $\operatorname{Sim} E$ acts transitively on E. Together with the above theorem this gives a classification of weakly-irreducible subgroups of $SO(1, n+1)_l$.

To the above decomposition of the Lie group $\operatorname{Sim} E$ corresponds the following decomposition of its Lie algebra $\mathrm{LA}(\operatorname{Sim} E)$:

$$\mathrm{LA}(\operatorname{Sim} E) = (\mathcal{A}^1 \oplus \mathfrak{so}(E)) \ltimes E,$$

where $\mathcal{A}^1 = \mathbb{R}$ is the Lie algebra of the Lie group A^1 and E is the Lie algebra of the Lie group E. We see that $\mathcal{A}^1$ commutes with $\mathfrak{so}(E)$, and E is a commutative ideal in $\mathrm{LA}(\operatorname{Sim} E)$.

Let $\mathcal{B}$ be a Lie algebra. We denote by $\mathcal{B}'$ the commutant of $\mathcal{B}$ and by $\mathfrak{z}(\mathcal{B})$ the center of $\mathcal{B}$. If $\mathcal{B} \subset \mathfrak{so}(n)$, then $\mathcal{B}$ is a compact Lie algebra and we have $\mathcal{B} = \mathcal{B}' \oplus \mathfrak{z}(\mathcal{B})$.

The Lie algebras of the Lie groups from the above theorem have the following forms (see [12]):

type 1. $\mathfrak{g}^{1,\mathcal{B}} = (\mathcal{A}^1 \oplus \mathcal{B}) \ltimes E$, where $\mathcal{B} \subset \mathfrak{so}(E)$ is a subalgebra;

type 2. $\mathfrak{g}^{2,\mathcal{B}} = \mathcal{B} \ltimes E$;

type 3. $\mathfrak{g}^{3,\mathcal{B},\varphi} = (\mathcal{B}' \oplus \{(\varphi(x) + x)|x \in \mathfrak{z}(\mathcal{B})\}) \ltimes E$, where $\varphi : \mathfrak{z}(\mathcal{B}) \to \mathcal{A}^1$ is a non-zero linear map;

type 4. $\mathfrak{g}^{4,k,\mathcal{B},\psi} = (\mathcal{B}' \oplus \{(\psi(x) + x)|x \in \mathfrak{z}(B)\}) \ltimes W$, where we have a non-trivial orthogonal decomposition $E = W \oplus U$ such that $\mathcal{B} \subset \mathfrak{so}(W)$, and $\psi : \mathfrak{z}(\mathcal{B}) \to U$ is a surjective linear map; $k = \dim W$.

It is convenient for us to extend the maps φ and ψ to $\mathcal{B}$ and to unify the Lie algebras of type 2 and 3 assuming that $\varphi = 0$ for the Lie algebras of type 2. We obtain the following

Theorem 2.4. *Let $G \subset \mathrm{Sim}\, E$ be a transitively acting connected subgroup. Then the Lie algebra of G belongs to one of the following types*

type $\mathcal{A}^1$. $\mathfrak{g}^{\mathcal{B},\mathcal{A}^1} = (\mathcal{A}^1 \oplus \mathcal{B}) \ltimes E$, *where $\mathcal{B} \subset \mathfrak{so}(E)$ is a subalgebra;*

type φ. $\mathfrak{g}^{\mathcal{B},\varphi} = \{\varphi(x) + x|x \in \mathcal{B}\} \ltimes E$, *where $\varphi : \mathcal{B} \to \mathcal{A}^1$ is a linear map with $\varphi|_{\mathcal{B}'} = 0$;*

type ψ. $\mathfrak{g}^{\mathcal{B},\psi,k} = \{\psi(x) + x|x \in \mathcal{B}\}) \ltimes W$, *where we have a non-trivial orthogonal decomposition $E = W \oplus U$ such that $\mathcal{B} \subset \mathfrak{so}(W)$, and $\psi : \mathcal{B} \to U$ is a surjective linear map with $\psi|_{\mathcal{B}'} = 0$; $k = \dim W$.*

For each Lie algebra $\mathfrak{g}$ from Theorem 2.4 we call $\mathcal{B} \subset \mathfrak{so}(E)$ *the orthogonal part of* $\mathfrak{g}$.

2.3 Similarity transformations of the Heisenberg spaces

In this section we explain notation from [14], which we will use later.

Let $\tilde{E}$ be a complex vector space of dimension n endowed with a Hermitian metric g.

By definition, *the Heisenberg space associated to* n is the direct sum $\mathcal{H}_n = \tilde{E} \oplus \mathbb{R}$. The line $\mathbb{R}$ is called the *vertical axis*.

We consider $\mathcal{H}_n$ also as a group with respect to the operation

$$(z, u) \cdot (w, v) = (z + w, u + v + 2\,\mathrm{Im}\, g(z, w)),$$

where $z, w \in \tilde{E}$ and $u, v \in \mathbb{R}$. The group $\mathcal{H}_n$ is nilpotent and $\mathbb{R} \subset \mathcal{H}_n$ is a normal subgroup.

We consider the action of the group $\mathcal{H}_n$ on itself by left translations. These transformations are called *the Heisenberg translations.*

The unitary group $U(\tilde{E})$ acts on $\mathcal{H}_n$ by

$$A : (z, u) \mapsto (Az, u),$$

where $A \in U(\tilde{E})$. These transformations are called *the Heisenberg rotations about the vertical axis.*

The group $\mathbb{C}^*$ of non-zero complex numbers acts on $\mathcal{H}_n$ by

$$\lambda : (z, u) \mapsto (\lambda z, |\lambda|^2 u),$$

where $\lambda \in \mathbb{C}^*$. These transformations are called *the complex Heisenberg dilations about the origin.*

The intersection of the group of the Heisenberg rotations about the vertical axis and the group of the complex Heisenberg dilations about the origin is the group of scalar multiplications by unit complex numbers $\mathbb{T} = \mathbb{C}^* \cap U(\tilde{E})$. The groups $\mathbb{C}^*$ and $U(\tilde{E})$ generate the group $\mathbb{R}^+ \times U(\tilde{E})$, where $\mathbb{R}^+$ is the group of *the real Heisenberg dilations about the origin*, i.e. with $\lambda \in \mathbb{R}^+$.

All the above transformations generate *the Heisenberg similarity transformation group*

$$\operatorname{Sim} \mathcal{H}_n = (\mathbb{R}^+ \times U(\tilde{E})) \leftthreetimes \mathcal{H}_n,$$

where the subgroup $\mathcal{H}_n \subset \operatorname{Sim} \mathcal{H}_n$ is normal.

By definition, *the similarity transformation group of the Hermitian space $\tilde{E}$* is

$$\operatorname{Sim} \tilde{E} = (\mathbb{R}^+ \times U(\tilde{E})) \leftthreetimes \tilde{E},$$

where $\tilde{E} \subset \operatorname{Sim} \tilde{E}$ is a normal subgroup that consists of translations in $\tilde{E}$.

We have the natural projection

$$\pi : \operatorname{Sim} \mathcal{H}_n \to \operatorname{Sim} \tilde{E}.$$

The kernel of π is 1-dimensional and consists of the Heisenberg translations $(z, u) \mapsto (z, u + c)$, $c \in \mathbb{R}$.

To the above decomposition of the Lie group $\mathrm{Sim}\,\mathcal{H}_n$ corresponds the following decomposition of its Lie algebra

$$\mathrm{LA}(\mathrm{Sim}\,\mathcal{H}_n) = (\mathcal{A}^1 \oplus \mathfrak{u}(n)) \ltimes \mathrm{LA}(\mathcal{H}_n),$$

where $\mathcal{A}^1 = \mathbb{R}$ is the Lie algebra of the Lie group of the real Heisenberg dilations about the origin.

Let us denote by $\mathcal{C}$ the Lie algebra of the group of the Heisenberg translations $(0, u)$: $(w, v) \mapsto (w, u + v)$. Let $\tilde{E} \subset \mathrm{LA}(\mathcal{H}_n)$ be the tangent space to the submanifold $\tilde{E} \subset \mathrm{Sim}\,\tilde{E}$ at the unit. Then $\mathcal{C}$ is an ideal in $\mathrm{LA}(\mathrm{Sim}\,\mathcal{H}_n)$ and $\tilde{E}$ is a vector subspace of $\mathrm{LA}(\mathcal{H}_n)$ with $[\tilde{E}, \tilde{E}] = \mathcal{C}$. We have $\mathrm{LA}(\mathcal{H}_n) = \tilde{E} + \mathcal{C}$. Thus,

$$\mathrm{LA}(\mathrm{Sim}\,\mathcal{H}_n) = (\mathcal{A}^1 \oplus \mathfrak{u}(n)) \ltimes (\tilde{E} + \mathcal{C}).$$

For the Lie algebra $\mathrm{LA}(\mathrm{Sim}\,\tilde{E})$ of the Lie group $\mathrm{Sim}\,\tilde{E}$ we obtain the decomposition

$$\mathrm{LA}(\mathrm{Sim}\,E) = (\mathcal{A}^1 \oplus \mathfrak{u}(n)) \ltimes \tilde{E},$$

where $\mathcal{A}^1 = \mathbb{R}$ is the Lie algebra of the group of real dilations about the origin, $\tilde{E}$ is the Lie algebra of the Lie group $\tilde{E}$. We see that A^1 commutes with $\mathfrak{u}(n)$, and $\tilde{E}$ is an ideal in $\mathrm{LA}(\mathrm{Sim}\,E)$.

We denote the differential of the projection $\pi : \mathrm{Sim}\,\mathcal{H}_n \to \mathrm{Sim}\,\tilde{E}$ also by π. Obviously, the linear map

$$\pi : \mathrm{LA}(\mathrm{Sim}\,\mathcal{H}_n) \to \mathrm{LA}(\mathrm{Sim}\,\tilde{E})$$

is surjective with the 1-dimensional kernel $\mathcal{C}$.

2.4 The groups $U(1, n+1)_{\mathbb{C}p_1}$ and $U(1, n+1)_{<p_1,p_2>}$, their Lie algebras and examples

Let S be a complex vector space. Denote by $S_{\mathbb{R}}$ the real vector space underlying S and by J the complex structure on $S_{\mathbb{R}}$. The correspondence $S \mapsto (S_{\mathbb{R}}, J)$ gives an isomorphism of categories of complex vector spaces and real vector spaces with complex structures. For a real vector space S with a complex structure J we denote by $\tilde{S}$ the complex vector space given by (S, J).

Let S be a complex vector space. A subspace $S_1 \subset S_{\mathbb{R}}$ is called *complex* if $JS_1 = S_1$, where J is the complex structure on $S_{\mathbb{R}}$. A subspace $S_1 \subset S_{\mathbb{R}}$ is called *a real form* of S if $JS_1 \cap S_1 = \{0\}$ and $\dim_{\mathbb{R}} S_1 = \dim_{\mathbb{C}} S$.

Suppose that S is endowed with a pseudo-Hermitian metric g. For $x, y \in S_{\mathbb{R}}$ let $\eta(x,y) = \operatorname{Re} g(x,y)$. Then η is a metric on $S_{\mathbb{R}}$ and we have $\eta(Jx, Jy) = \eta(x,y)$ for all $x, y \in S_{\mathbb{R}}$, i.e. η is J-invariant. Conversely, for a given real vector space S with a complex structure J and an J-invariant non-degenerate metric η, let $g(z,w) = \eta(z,w) + i\eta(z, Jw)$ for all $z, w \in \tilde{S}$. Then g is a pseudo-Hermitian metric on $\tilde{S}$. This gives us an isomorphism of categories of pseudo-Hermitian spaces and real vector spaces endowed with complex structures and invariant non-degenerate metrics.

A vector subspace $L \subset S$ (resp. $\tilde{S}$) is called *degenerate* if the restriction of η (resp. g) to L is degenerate. A vector subspace $L \subset S$ (resp. $\tilde{S}$) is called *isotropic* if the restriction of η (resp. g) to L is zero.

Let S be a complex vector space of dimension $n+2$, where $n \geq 0$, and let g be a pseudo-Hermitian metric on S of signature $(1, n+1)$. By definition, the pseudo-unitary group $U(1, n+1)$ is the real Lie group of g-invariant automorphisms of S, i.e.

$$U(1, n+1) = \{f \in \operatorname{Aut}(S) | g(fz, fw) = g(z,w) \text{ for all } z, w \in S\}.$$

The corresponding Lie algebra consists of g-skew symmetric endomorphisms of S, i.e.

$$\mathfrak{u}(1, n+1) = \{\xi \in \operatorname{End}(S) | g(\xi z, w) + g(z, \xi w) = 0 \text{ for all } z, w \in S\}.$$

Consider the action of the group $U(1, n+1)$ on $S_{\mathbb{R}}$. Then

$$U(1, n+1) = \{f \in SO(2, 2n+2) | Jf = fJ\}$$

and

$$\mathfrak{u}(1, n+1) = \{\xi \in \mathfrak{so}(2, 2n+2) | [\xi, J] = 0\}.$$

Here $SO(2, 2n+2)$ is the Lie group of η-orthogonal automorphisms of $S_{\mathbb{R}}$ and $\mathfrak{so}(2, 2n+2)$ is the corresponding Lie algebra, which consists of η-skew symmetric endomorphisms of $S_{\mathbb{R}}$.

Let $\mathbb{R}^{2,2n+2}$ be a $2n+4$-dimensional real vector space endowed with a complex structure $J \in \operatorname{Aut} \mathbb{R}^{2,2n+2}$ and with a J-invariant metric η of signature $(2, 2n+2)$. Let $\mathbb{C}^{1,n+1}$ be the $n+2$-dimensional complex vector space given by $(\mathbb{R}^{2,2n+2}, J, \eta)$. Denote by g the pseudo-Hermitian metric on $\mathbb{C}^{1,n+1}$ of signature $(1, n+1)$ corresponding to η.

We say that a subgroup $G \subset U(1, n+1) \subset SO(n+2, \mathbb{C})$ *acts weakly-irreducibly on* $\mathbb{C}^{1,n+1}$ *if it does not preserve any non-degenerate proper subspace of* $\mathbb{C}^{1,n+1}$. *We say that a subalgebra* $\mathfrak{g} \subset \mathfrak{u}(1, n+1) \subset \mathfrak{so}(n+2, \mathbb{C})$ *is weakly-irreducible if it does not preserve any non-degenerate proper subspace of* $\mathbb{C}^{1,n+1}$.

Proposition 2.1. *If a subgroup $G \subset U(1,n+1)$ acts weakly-irreducibly on $\mathbb{R}^{2,2n+2}$, then G acts weakly-irreducibly on $\mathbb{C}^{1,n+1}$.*

Proof. Suppose that $G \subset U(1,n+1)$ acts weakly-irreducibly on $\mathbb{R}^{2,2n+2}$ and G preserves a non-degenerate proper subspace $L \subset \mathbb{C}^{1,n+1}$. Hence G preserves the subspace $L_{\mathbb{R}} \subset \mathbb{R}^{2,2n+2}$. We claim that $L_{\mathbb{R}}$ is non-degenerate. Indeed, let $x \in L_{\mathbb{R}} \cap (L_{\mathbb{R}})^{\perp_\eta}$. For any $y \in L$ we have $g(x,y) = \eta(x,y) + i\eta(x,Jy) = 0$, since $JL = L$. Hence, $x \in L \cap L^{\perp_g}$ and $x = 0$. Thus the subspace $L_{\mathbb{R}}$ is non-degenerate and we have a contradiction. $\Box$

The converse to Proposition 2.1 is not true, see Example 2.3 below.

For the group $U(1,n+1)$ we have the local decomposition $U(1,n+1) = SU(1,n+1)\cdot\mathbb{T}$, where $SU(1,n+1) = U(1,n+1) \cap SL(2+n,\mathbb{C})$ and $\mathbb{T}$ is the center of $U(1,n+1)$, which is the 1-dimensional subgroup generated by the complex structure $J \in U(1,n+1)$.

We fix a basis $p_1, e_1, ..., e_n, q_1$ of $\mathbb{C}^{1,n+1}$ such that the Gram matrix of g has the form

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & E_n & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

where E_n is the n-dimensional identity matrix. Let $\tilde{E} \subset \mathbb{C}^{1,n+1}$ be the vector subspace spanned by $e_1, ..., e_n$. We will consider $\tilde{E}$ as Hermitian space with the metric $g|_{\tilde{E}}$.

Denote by $U(1,n+1)_{\mathbb{C}p_1}$ the Lie subgroup of $U(1,n+1)$ acting on $\mathbb{C}^{1,n+1}$ and preserving the complex isotropic line $\mathbb{C}p_1$. Since $J \in U(1,n+1)_{\mathbb{C}p_1}$, we have the decomposition $U(1,n+1)_{\mathbb{C}p_1} = SU(1,n+1)\cdot\mathbb{T}$, where $SU(1,n+1)_{\mathbb{C}p_1} = U(1,n+1)_{\mathbb{C}p_1} \cap SL(2+n,\mathbb{C})$.

The Lie algebra $\mathfrak{u}(1,n+1)_{\mathbb{C}p_1}$ of the Lie group $U(1,n+1)_{\mathbb{C}p_1}$ can be identified with the following matrix algebra

$$\mathfrak{u}(1,n+1)_{\mathbb{C}p_1} = \left\{ \left. \begin{pmatrix} a & -\overline{z}^t & ic \\ 0 & A & z \\ 0 & 0 & -\overline{a} \end{pmatrix} \right| a \in \mathbb{C},\, c \in \mathbb{R},\, z \in \tilde{E},\, A \in \mathfrak{u}(n) \right\}.$$

Here $\mathfrak{u}(n)$ is the unitary Lie algebra of the Hermitian space $\tilde{E}$. We identify the above matrix with the quadruple (a, A, z, c) and define the following vector subspaces of $\mathfrak{u}(1,n+1)_{\mathbb{C}p_1}$:

$$\mathcal{A} = \{(a,0,0,0)|a \in \mathbb{C}\}, \quad \mathcal{N} = \{(0,0,z,0)|z \in \tilde{E}\} \quad \text{and} \quad \mathcal{C} = \{(0,0,0,c)|c \in \mathbb{R}\}.$$

We consider $\mathfrak{u}(n)$ as a subalgebra of $\mathfrak{u}(1,n+1)_{\mathbb{C}p_1}$ with the obvious inclusion. Note that $\mathcal{C}$ is a commutative ideal, which commutes with $\mathfrak{u}(n)$ and $\mathcal{N}$, and $\mathcal{A}$ is a commutative subalgebra, which commutes with $\mathfrak{u}(n)$. For $a \in \mathbb{C}$, $c \in \mathbb{R}$, $z, w \in \tilde{E}$ and $A \in \mathfrak{u}(n)$ we

obtain

$$[(a,0,0,0),(0,0,z,c)] = (0,0,\overline{a}z, 2\operatorname{Re} a\cdot c), \quad [(0,A,0,0),(0,0,z,0)] = (0,0,Az,0)$$

and

$$[(0,0,z,0),(0,0,w,0)] = (0,0,0,-2\operatorname{Im} g(z,w)).$$

We obtain the decomposition

$$\mathfrak{u}(1,n+1)_{\mathbb{C}p_1} = (\mathcal{A}\oplus\mathfrak{u}(n))\ltimes(\mathcal{N}+\mathcal{C}).$$

For the Lie algebra $\mathfrak{su}(1,n+1)_{\mathbb{C}p_1}$ of the Lie group $SU(1,n+1)_{\mathbb{C}p_1}$ we have $\mathfrak{su}(1,n+1)_{\mathbb{C}p_1} = \mathfrak{u}(1,n+1)_{\mathbb{C}p_1}\cap\mathfrak{su}(1,n+1) = \{\xi\in\mathfrak{u}(1,n+1)_{\mathbb{C}p_1}\,|\,\operatorname{tr}_{\mathbb{C}}\xi = 0\} = \{(a,A,z,c)\in\mathfrak{u}(1,n+1)_{\mathbb{C}p_1}|a-\bar{a}+\operatorname{tr}_{\mathbb{C}}A = 0\}$ and $\mathfrak{u}(1,n+1)_{\mathbb{C}p_1} = \mathfrak{su}(1,n+1)_{\mathbb{C}p_1}\oplus\mathbb{R}J$.

Let $\mathfrak{g}\subset\mathfrak{u}(1,n+1)\subset\mathfrak{so}(n+2,\mathbb{C})$ be a weakly-irreducible and not irreducible subalgebra. Then $\mathfrak{g}$ preserves a non-degenerate proper subspace $L\subset\mathbb{C}^{1,n+1}$. Hence $\mathfrak{g}$ preserves the orthogonal complement $L^{\perp_g}$ and the intersection $L\cap L^{\perp_g}$, which is an isotropic complex line. Hence $\mathfrak{g}$ is conjugated to a weakly-irreducible subalgebra of $\mathfrak{u}(1,n+1)_{\mathbb{C}p_1}$.

Now we consider the real vector space $\mathbb{R}^{2,2n+2}$. Let $p_2 = Jp_1$, $f_1 = Je_1$,..., $f_n = Je_n$ and $q_2 = Jq_1$. Consider the basis p_1, p_2, e_1,...,e_n, f_1,...,f_n, q_1, q_2 of the vector space $\mathbb{R}^{2,2n+2}$. With respect to this basis the Gram matrix of the metric η and the complex structure J have the forms

$$\begin{pmatrix} 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&E_{2n}&0&0\\ 1&0&0&0&0\\ 0&1&0&0&0\end{pmatrix} \text{ and } \begin{pmatrix} 0&-1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&-E_n&0&0\\ 0&0&E_n&0&0&0\\ 0&0&0&0&0&-1\\ 0&0&0&0&1&0\end{pmatrix}, \text{ respectively.}$$

We consider the vector space $E = \operatorname{span}_{\mathbb{R}}\{e_1,...,e_n,f_1,...,f_n\}\subset\mathbb{R}^{2,2n+2}$ as an Euclidian space with the metric $\eta|_E$. Let $E^1 = \operatorname{span}_{\mathbb{R}}\{e_1,...,e_n\}$ and $E^2 = \operatorname{span}_{\mathbb{R}}\{f_1,...,f_n\}$.

We denote by $U(1,n+1)_{<p_1,p_2>}$ the subgroup of $U(1,n+1)$ acting on $\mathbb{R}^{2,2n+2}$ and preserving the isotropic 2-dimensional vector subspace $\mathbb{R}p_1\oplus\mathbb{R}p_2\subset\mathbb{R}^{2,2n+2}$. The group $U(1,n+1)_{<p_1,p_2>}$ is just $U(1,n+1)_{\mathbb{C}p_1}$ acting on $\mathbb{R}^{2,2n+2}$. Denote by $SU(1,n+1)_{<p_1,p_2>}$ the group $SU(1,n+1)_{\mathbb{C}p_1}$ acting on $\mathbb{R}^{2,2n+2}$. We have $U(1,n+1)_{<p_1,p_2>} = SU(1,n+1)_{<p_1,p_2>}\cdot\mathbb{T}$.

The Lie algebra $\mathfrak{u}(1,n+1)_{<p_1,p_2>}$ of the Lie group $U(1,n+1)_{<p_1,p_2>}$ can be identified with the matrix algebra as in Section 2.1. The correspondence between $\mathfrak{u}(1,n+1)_{\mathbb{C}p_1}$ and $\mathfrak{u}(1,n+1)_{<p_1,p_2>}$ is given by the identities $a = a_1+ia_2$, $A = B+iC$ and $z = z_1+iz_2$.

Now we consider some examples (we use denotation of Section 2.1).

Example 2.1. *The subalgebra* $\mathfrak{g} = \mathcal{N}^1 + \mathcal{C} \subset \mathfrak{u}(1, n+1)_{<p_1,p_2>}$ *is weakly-irreducible.*

Proof. Suppose that $\mathfrak{g}$ preserves a proper vector subspace $L \subset \mathbb{R}^{2,2n+2}$. Let $v = (a_1, a_2, \alpha, \beta, b_1, b_2) \in L$, where we use the decomposition

$$\mathbb{R}^{2,2n+2} = \mathbb{R}p_1 \oplus \mathbb{R}p_2 \oplus \mathrm{span}_{\mathbb{R}}\{e_1, ..., e_n\} \oplus \mathrm{span}_{\mathbb{R}}\{f_1, ..., f_n\} \oplus \mathbb{R}q_1 \oplus \mathbb{R}q_2.$$

Applying an element $(0, 0, 0, 0, z_1, 0, 0) \in \mathcal{N}^1 \subset \mathfrak{g}$ with $z_1^t z_1 = 1$ twice to v, we see that $b_1p_1 + b_2p_2 \in L$. Applying the element $(0, 0, 0, 0, 0, 0, 1) \in \mathcal{C} \subset \mathfrak{g}$ to v, we get $-b_2p_1 + b_1p_2 \in L$. Hence if the projection of L to $\mathbb{R}q_1 \oplus \mathbb{R}q_2$ is non-zero, then L contains $\mathbb{R}p_1 \oplus \mathbb{R}p_2$ and the projection of $L^{\perp_\eta}$ to $\mathbb{R}q_1 \oplus \mathbb{R}q_2$ is zero, moreover, $L^{\perp_\eta}$ is also preserved. Thus we can assume that the projection of L to $\mathbb{R}q_1 \oplus \mathbb{R}q_2$ is zero. Let $v = (a_1, a_2, \alpha, \beta, 0, 0) \in L$. If $\alpha \neq 0$, then applying the element $(0, 0, 0, 0, \alpha, 0, 0) \in \mathcal{N}^1 \subset \mathfrak{g}$ to v, we see that $0 \neq \alpha^t\alpha p_1 + \alpha^t\beta p_2 \in L$, hence L is degenerate. The similar statement is for $\beta \neq 0$. If $\alpha = \beta = 0$ for all $v \in L$, then L is degenerate. $\Box$

Example 2.2. *The subalgebra* $\mathfrak{g} = \mathcal{N}^1 \subset \mathfrak{u}(1, n+1)_{<p_1,p_2>}$ *is not weakly-irreducible. The subalgebra* $\mathfrak{g} = \mathcal{N}^1 \subset \mathfrak{u}(1, n+1)_{\mathbb{C}p_1}$ *is weakly-irreducible.*

Proof. Suppose that $\mathfrak{g}$ preserves a proper subspace $L \subset \mathbb{C}^{1,n+1}$. Let $v = (a_1, \alpha, b_1) \in L$ (here we use the decomposition $\mathbb{C}^{1,n+1} = \mathbb{C}p_1 \oplus \mathrm{span}_{\mathbb{C}}\{e_1, ..., e_n\} \oplus \mathbb{C}q_1$). Applying an element $(0, 0, z_1, 0) \in \mathcal{N}^1$ with $\bar{z}_1^t z_1 = 1$ twice to v, we see that $b_1p_1 \in L$. As in Example 2.1, we can assume that the projection of $\mathfrak{g}$ to $\mathbb{C}q_1$ is zero and show that L is degenerate.

The vector subspaces $\mathrm{span}_{\mathbb{R}}\{p_1 + p_2, e_1 + f_1, ..., e_n + f_n, q_1 + q_2\} \subset \mathbb{R}^{2,2n+2}$ and $\mathrm{span}_{\mathbb{R}}\{p_1 - p_2, e_1 - f_1, ..., e_n - f_n, q_1 - q_2\} \subset \mathbb{R}^{2,2n+2}$ are non-degenerate and preserved by $\mathfrak{g}$. $\Box$

Example 2.3. *The subalgebra* $\mathfrak{g} = \mathcal{N}^1 \oplus \mathbb{R}J \subset \mathfrak{u}(1, n+1)_{<p_1,p_2>}$ *is weakly-irreducible.*

Proof. Suppose that $\mathfrak{g}$ preserves a proper vector subspace $L \subset \mathbb{R}^{2,2n+2}$. Let $v = (a_1, a_2, \alpha, \beta, b_1, b_2) \in L$. As in Example 2.1, we see that $b_1p_1 + b_2p_2 \in L$. Applying the element $J \in \mathfrak{g}$ to $b_1p_1 + b_2p_2$, we get $-b_2p_1 + b_1p_2 \in L$. The end of the proof is as in Example 2.1. $\Box$

Let A^1, A^2, N^1, N^2 and C be the connected Lie subgroups of $U(1, n+1)_{\mathbb{C}p_1}$ corresponding to the subalgebras $\mathcal{A}^1$, $\mathcal{A}^2$, $\mathcal{N}^1$, $\mathcal{N}^2$ and $\mathcal{C}$ of the Lie algebra $\mathfrak{u}(1, n+1)_{\mathbb{C}p_1}$. These groups can be identified with the following groups of matrices

$$A^1 = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & \mathrm{id} & 0 \\ 0 & 0 & \frac{1}{a_1} \end{pmatrix} \middle| \begin{array}{c} a_1 \in \mathbb{R}, \\ a_1 > 0 \end{array} \right\}, \qquad A^2 = \left\{ \begin{pmatrix} e^{ia_2} & 0 & 0 \\ 0 & \mathrm{id} & 0 \\ 0 & 0 & e^{ia_2} \end{pmatrix} \middle| a_2 \in \mathbb{R} \right\},$$

$$N^1 = \left\{ \begin{pmatrix} 1 & -z_1^t & -\frac{1}{2} z_1^t z_1 \\ 0 & \mathrm{id} & z_1 \\ 0 & 0 & 1 \end{pmatrix} \middle| z_1 \in E^1 \right\}, \quad N^2 = \left\{ \begin{pmatrix} 1 & z_2^t & \frac{1}{2} z_2^t z_2 \\ 0 & \mathrm{id} & z_2 \\ 0 & 0 & 1 \end{pmatrix} \middle| z_2 \in E^2 \right\},$$

$$C = \left\{ \begin{pmatrix} 1 & 0 & ic \\ 0 & \mathrm{id} & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| c \in \mathbb{R} \right\}.$$

We consider the group $U(n)$ as a Lie subgroup of $U(1, n+1)_{\mathbb{C}p_1}$ with the inclusion

$$A \in U(n) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix} \in U(1, n+1)_{\mathbb{C}p_1}.$$

We have the decomposition $U(1, n+1)_{\mathbb{C}p_1} = (A^1 \times A^2 \times U(n)) \ltimes (N \cdot C)$.

2.5 Action of the group $U(1, n+1)_{\mathbb{C}p_1}$ on the boundary of the complex hyperbolic space

As above, let $\mathbb{C}^{1,n+1}$ be a complex vector space of dimension $n+2$ and g be a pseudo-Hermitian metric on $\mathbb{C}^{1,n+1}$ of signature $(1, n+1)$. A complex line $l \subset \mathbb{C}^{1,n+1}$ is called *negative* if $g(z, z) < 0$ for all $z \in l \backslash \{0\}$. The $n+1$-dimensional *complex hyperbolic* space $\mathbf{H}_{\mathbb{C}}^{n+1}$ is the subset of the projective space $\mathbb{P}\mathbb{C}^{1,n+1}$ consisting of all negative lines.

The *boundary* $\partial \mathbf{H}_{\mathbb{C}}^{n+1}$ of the complex hyperbolic space $\mathbf{H}_{\mathbb{C}}^{n+1}$ is the subset of the projective space $\mathbb{P}\mathbb{C}^{1,n+1}$ consisting of all complex isotropic lines.

We identify $\partial \mathbf{H}_{\mathbb{C}}^{n+1}$ with a $2n+1$-dimensional real sphere in the following way. Let $p_1, e_1, ..., e_n, q_1$ by the basis of $\mathbb{C}^{1,n+1}$ as above. We also consider the basis $e_0, e_1, ..., e_n, e_{n+1}$, where $e_0 = \frac{\sqrt{2}}{2}(p_1 - q_1)$ and $e_{n+1} = \frac{\sqrt{2}}{2}(p_1 + q_1)$. With respect to this basis the Gram matrix of g has the form

$$\begin{pmatrix} -1 & 0 \\ 0 & E_{n+1} \end{pmatrix}.$$

Consider the vector subspace $\tilde{E}_1 = \tilde{E} \oplus \mathbb{C}e_{n+1} \subset \tilde{E}$. Each isotropic line intersects the affine subspace $e_0 + \tilde{E}_1$ at an unique point and we see that

$$(e_0 + \tilde{E}_1) \cap \{z \in \mathbb{C}^{1,n+1} | g(z,z) = 0\} = \\ \{z \in \mathbb{R}^{2,2n+2} | x_0 = 1,\ y_0 = 0,\ x_1^2 + y_1^2 + \cdots + x_{n+1}^2 + y_{n+1}^2 = 1\}$$

is a $2n+1$-dimensional unite sphere S^{2n+1}. Here $(x_0 + iy_0, ..., x_{n+1} + iy_{n+1})$ are the coordinates of a point z with respect to the basis $e_0, e_1, ..., e_n, e_{n+1}$.

Let $G \subset U(1, n+1)_{\mathbb{C}p_1}$ be a subgroup. We identify $\partial \mathbf{H}^{n+1}_{\mathbb{C}} \backslash \{\mathbb{C}p_1\}$ with a Heisenberg space $\mathcal{H}_n$ and consider an action of G on $\mathcal{H}_n$. For this let $E_2 = \tilde{E} \oplus i\mathbb{R}e_{n+1}$ and $p_0 = \mathbb{C}p_1 \cap (e_0 + \tilde{E}_1) = \sqrt{2}p_1$. Denote by s_0 the stereographic projection $s_0 : S^{2n+1} \backslash \{p_0\} \to E_2$ (here we identify E_2 and $e_0 + E_2$). Note that E_2 is just a Heisenberg space $\mathcal{H}_n$ with the vertical axis $i\mathbb{R}e_{n+1}$. Any $f \in U(1, n+1)_{\mathbb{C}p_1}$ takes complex isotropic lines to complex isotropic lines, hence it acts on $S^{2n+1} \backslash \{p_0\} = \partial \mathbf{H}^{n+1}_{\mathbb{C}} \backslash \{\mathbb{C}p_1\}$, and we get the transformation $\Gamma(f) = s_0 \circ f \circ s_0^{-1}$ of $E_2 = \mathcal{H}_n$. Moreover, we will see that this transformation is a Heisenberg similarity transformation.

Now we consider the basis $p_1, e_1, ..., e_n, q_1$ and the matrices of elements of $U(1, n+1)_{\mathbb{C}p_1}$ with respect to this basis. Computations show that

for $a_1 \in \mathbb{R}$, $a_1 > 0$, the element $\begin{pmatrix} a_1 & 0 & 0 \\ 0 & \mathrm{id} & 0 \\ 0 & 0 & \frac{1}{a_1} \end{pmatrix}$ corresponds to the real Heisenberg dilation $(z, iu) \mapsto (a_1 z, a_1^2 iu)$,

where $z \in \tilde{E}$ and $u \in \mathbb{R}$;

for $a_2 \in \mathbb{R}$ the element $\begin{pmatrix} e^{ia_2} & 0 & 0 \\ 0 & \mathrm{id} & 0 \\ 0 & 0 & e^{ia_2} \end{pmatrix}$ corresponds to the complex Heisenberg dilation $(z, iu) \mapsto (e^{-ia_2} z, iu)$;

for $A \in U(n)$ the element $\begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix}$ corresponds the Heisenberg rotation $(z, iu) \mapsto (Az, iu)$;

for $z_1 \in E^1$, $z_2 \in E^2$ and $c \in \mathbb{R}$ the elements

$$\begin{pmatrix} 1 & -z_1^t & -\frac{1}{2} z_1^t z_1 \\ 0 & \mathrm{id} & z_1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & z_2^t & \frac{1}{2} z_2^t z_2 \\ 0 & \mathrm{id} & z_2 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & ic \\ 0 & \mathrm{id} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

correspond to the Heisenberg translations

$(z, iu) \mapsto (z + z_1, iu + i2 \operatorname{Im} g(z, z_1))$, $\quad (z, iu) \mapsto (z + z_2, iu + i2 \operatorname{Im} g(z, z_2))$

and $(z, iu) \mapsto (z, iu + ic)$, respectively.

Thus we have a surjective Lie group homomorphism

$$\Gamma : U(1,n+1)_{\mathbb{C}p_1} \to \operatorname{Sim} \mathcal{H}_n.$$

The kernel of Γ is the center $\mathbb{T}$ of $U(1,n+1)_{\mathbb{C}p_1}$, hence the restriction

$$\Gamma|_{SU(1,n+1)_{\mathbb{C}p_1}} : SU(1,n+1)_{\mathbb{C}p_1} \to \operatorname{Sim} \mathcal{H}_n$$

is a Lie group isomorphism.

We say that an affine subspace $L \subset E$ is *complex* if the corresponding vector subspace is complex, otherwise we say that L is *non-complex*.

Let $\pi : \operatorname{Sim} \mathcal{H}_n \to \operatorname{Sim} \tilde{E}$ be the obvious projection. The homomorphism π is surjective and its kernel is 1-dimensional and consists of the Heisenberg translations $(z, iu) \mapsto (z, iu + ic)$.

Theorem 2.5. *Let $G \subset U(1,n+1)_{\mathbb{C}p_1}$ be a weakly-irreducible subgroup. Then*

(1) *The subgroup $\pi(\Gamma(G)) \subset \operatorname{Sim} \tilde{E}$ does not preserve any proper complex affine subspace of E.*

(2) *If $\pi(\Gamma(G)) \subset \operatorname{Sim} \tilde{E}$ preserves a proper non-complex affine subspace $L \subset E$, then the minimal complex affine subspace of $\tilde{E}$ containing L is $\tilde{E}$.*

Proof. (1) First we prove that $\pi(\Gamma(G)) \subset \operatorname{Sim} \tilde{E}$ does not preserve any proper complex vector subspace of E. Suppose that $\pi(\Gamma(G))$ preserves a proper complex vector subspace $L \subset \tilde{E}$. Then we have $\pi(\Gamma(G)) \subset (\mathbb{R}^+ \times U(L) \times U(L^{\perp_g})) \ltimes U$, where $L^{\perp_g}$ is the orthogonal complement to L in $\tilde{E}$. We see that the group G preserves the proper non-degenerate vector subspace $L^{\perp_g} \subset \mathbb{C}^{1,n+1}$.

Suppose that $\pi(\Gamma(G)) \subset \operatorname{Sim} \tilde{E}$ preserves a proper complex affine subspace $L \subset E$. Let $w \in L$ and let $L_0 = -w + L$ be the corresponding to L complex vector subspace of $\tilde{E}$. Let $f = \begin{pmatrix} 1 & -\overline{w}^t & -\frac{1}{2}\overline{w}^t w \\ 0 & \mathrm{id} & w \\ 0 & 0 & 1 \end{pmatrix} \in U(1,n+1)_{\mathbb{C}p_1}$. Consider the subgroup $G_1 = f^{-1} G f \subset U(1,n+1)_{\mathbb{C}p_1}$. We have $\pi(\Gamma(G_1)) = -w \cdot \pi(\Gamma(G)) \cdot w$ and $\pi(\Gamma(G_1))$ preserves the complex vector subspace $L_0 \subset \tilde{E}$. Hence G_1 preserves the complex vector subspace $L_0^{\perp_g} \subset \mathbb{C}^{1,n+1}$, which is non-degenerate. Thus G preserves the proper complex subspace $f(L_0^{\perp_g}) \subset \mathbb{C}^{1,n+1}$, which is also non-degenerate. This gives us a contradiction.

(2) Suppose that $\pi(\Gamma(G)) \subset \operatorname{Sim} \tilde{E}$ preserves a non-complex affine subspace $L \subset E$. By the same argument, G preserves the proper complex subspace $(\operatorname{span}_{\mathbb{C}} L_0)^{\perp_g} \subset \mathbb{C}^{1,n+1}$, which is non-degenerate. Hence, $\operatorname{span}_{\mathbb{C}} L_0 = \tilde{E}$. This proves the theorem. $\square$

2.6 Proof of Theorem 2.1

1) First consider the case $n = 0$. There are four subalgebras of $\mathfrak{su}(1,1)_{<p_1,p_2>} = \mathcal{A}^1 \ltimes \mathcal{C}$:

$$\mathcal{A}^1 \ltimes \mathcal{C}, \qquad \mathcal{C}, \qquad \mathcal{A}^1, \qquad \{(c\gamma, c) \in \mathcal{A}^1 \ltimes \mathcal{C} | c \in \mathbb{R}\}, \text{ where } \gamma \in \mathbb{R}, \gamma \neq 0.$$

The last subalgebra preserves the non-degenerate proper subspace $\operatorname{span}\{p_1 + p_2 + \gamma q_1 + \gamma q_2, 2p_2 - \gamma q_1 - \gamma q_2\} \subset \mathbb{R}^{2,2}$. The subalgebra $\mathcal{A}^1 \subset \mathfrak{su}(1,1)_{<p_1,p_2>}$ preserves the non-degenerate proper subspace $\mathbb{R}p_1 \oplus \mathbb{R}q_1 \subset \mathbb{R}^{2,2}$.

We claim that the subalgebra $\mathcal{C} \subset \mathfrak{su}(1,1)_{<p_1,p_2>}$ is weakly-irreducible. Indeed, suppose that $\mathcal{C}$ preserves a proper subspace $L \subset \mathbb{R}^{2,2}$. We may assume that $\dim L = 1$ or $\dim L = 2$ (if $\dim L = 3$, then we consider $L^\perp$). Let $\alpha_1 p_1 + \alpha_2 p_2 + \beta_1 q_1 + \beta_2 q_2 \in L$ be a non-zero vector. Applying the element $(0,1) \in \mathcal{C}$, we get $\beta_1 p_2 - \beta_2 p_1 \in L$. If $\dim L = 1$, then $\beta_1 = \beta_2 = 0$ and $L = \mathbb{R}(\alpha_1 p_1 + \alpha_2 p_2)$. If $\dim L = 2$, then $L = \operatorname{span}\{\alpha_1 p_1 + \alpha_2 p_2 + \beta_1 q_1 + \beta_2 q_2, \beta_1 p_2 - \beta_2 p_1\}$. In both cases L is degenerate.

Thus the subalgebra $\mathcal{C} \subset \mathfrak{su}(1,1)_{<p_1,p_2>}$ is weakly-irreducible. Hence the subalgebra $\mathfrak{su}(1,1)_{<p_1,p_2>} = \mathcal{A}^1 \ltimes \mathcal{C}$ is also weakly-irreducible. Part 1) of Theorem 2.1 is proved.

2) Suppose that $n \geq 1$. Theorem 2.5 shows us that we must find all connected Lie subgroups $F \subset \operatorname{Sim} \tilde{E}$ that satisfy the conclusion of Theorem 2.5. Then for each subgroup F find all connected subgroups $G \subset SU(1, n+1)_{<p_1,p_2>}$ with $\pi(\Gamma(G)) = F$ and check which of these groups act weakly-irreducibly on $\mathbb{R}^{2,2n+2}$. Since all groups are connected, we may do some steps in terms of their Lie algebras.

Step 1. First we describe non-complex vector subspaces $L \subset E$ with $\operatorname{span}_{\mathbb{C}} L = \tilde{E}$. Let $L \subset E$ be such a subspace and $L_0 = L \cap JL$. Let L_1 be the orthogonal complement to L_0 in L. Obviously, L_0 is a complex subspace and $L_1 \cap JL_1 = \{0\}$. Since $\operatorname{span}_{\mathbb{C}} L = \tilde{E}$, we see that $\operatorname{span}_{\mathbb{C}} L_1$ is the orthogonal complement to L_0 in $\tilde{E}$. Thus L_1 is a real form of the complex vector space $\operatorname{span}_{\mathbb{C}} L_1$. We choose the above basis $e_1, ..., e_n, f_1, ..., f_n$ in such a way that $L_0 = \operatorname{span}_{\mathbb{R}}\{e_1, ..., e_m, f_1, ..., f_m\}$ and $L_1 = \operatorname{span}_{\mathbb{R}}\{e_{m+1}, ..., e_n\}$, where $m = \dim_{\mathbb{C}} L_0 = \dim_{\mathbb{R}} L - n$ and $0 \leq m \leq n$. We have three cases:

Case $m = n$. $L = \tilde{E}$ is the whole space;

Case $m = 0$. $L = \operatorname{span}_{\mathbb{R}}\{e_1, ..., e_n\}$ is a real form of $\tilde{E}$;

Case $0 < m < n$. $L = \operatorname{span}_{\mathbb{R}}\{e_1, ..., e_m, f_1, ..., f_m\} \oplus \operatorname{span}_{\mathbb{R}}\{e_{m+1}, ..., e_n\}$.

Step 2. Now we describe Lie algebras $\mathfrak{f}$ of connected Lie subgroups $F \subset \operatorname{Sim} \tilde{E}$ preserving L. Without loss of generality, we may assume that the Lie group F does not preserve any

proper affine subspace of L, i.e. F acts irreducibly on L, hence F acts transitively on L (Theorem 2.2). And we can describe all such groups and their Lie algebras.

Case $m = n$**.** The Lie algebras corresponding to the transitive similarity transformation groups of type $\mathcal{A}^1$ and φ are

$\mathfrak{f}^{n,\mathcal{B},\mathcal{A}^1} = (\mathcal{A}^1 \oplus \mathcal{B}) \ltimes E$, where $\mathcal{B} \subset \mathfrak{u}(n)$ is a subalgebra;

$\mathfrak{f}^{n,\mathcal{B},\varphi} = \{x + \varphi(x) | x \in \mathcal{B}\}) \ltimes E$, where $\varphi : \mathcal{B} \to \mathcal{A}^1$ is a linear map with $\varphi|_{\mathcal{B}'} = 0$.

Now consider the Lie algebras of type ψ. Here we have an orthogonal decomposition $E = W \oplus U$. Let $W_1 = W \cap JW$, $U_1 = U \cap JU$ and let W_2 and U_2 be the orthogonal complements to W_1 and U_1 in W and U, respectively. Obviously, W_1 and U_1 are complex subspaces of E, and W_2 and U_2 are mutually orthogonal real forms of the complex vector space $\operatorname{span}_{\mathbb{C}} W_2 = \operatorname{span}_{\mathbb{C}} U_2$. We obtain the orthogonal decomposition of E: $E = W_1 \oplus U_1 \oplus W_2 \oplus U_2$. Let $k = \dim_{\mathbb{C}} W_1$ and $l = \dim_{\mathbb{C}}(W_1 \oplus U_1)$. We choose the basis $e_1, ..., e_n, f_1, ..., f_n$ in such a way that

$W_1 = \operatorname{span}_{\mathbb{R}}\{e_1, ..., e_k, f_1, ..., f_k\}$, $U_1 = \operatorname{span}_{\mathbb{R}}\{e_{k+1}, ..., e_l, f_{k+1}, ..., f_l\}$ and $W_2 = \operatorname{span}_{\mathbb{R}}\{e_{l+1}, ..., e_n\}$. Then $U_2 = JW_2 = \operatorname{span}_{\mathbb{R}}\{f_{l+1}, ..., f_n\}$.

The orthogonal part of a Lie algebra of type ψ is contained in $\mathfrak{so}(W) \cap \mathfrak{u}(n)$. Suppose $A = \left(\begin{smallmatrix} B & -C \\ C & B \end{smallmatrix}\right) \in \mathfrak{so}(W) \cap \mathfrak{u}(n)$, then with respect to the decomposition

$$E = E^1_{1,\dots,k} \oplus E^1_{k+1,\dots,l} \oplus E^1_{l+1,\dots,n} \oplus E^2_{1,\dots,k} \oplus E^2_{k+1,\dots,l} \oplus E^2_{l+1,\dots,n}$$

the matrix A has the form $\begin{pmatrix} B_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ C_1 & 0 & 0 & B_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. Consequently, $\mathfrak{so}(W) \cap \mathfrak{u}(n) = \mathfrak{u}(W_1) = \mathfrak{u}(k)$.

Thus the Lie algebras corresponding to transitively acting groups of type ψ have the form

$$\mathfrak{f}^{n,\mathcal{B},\psi,k,l} = \{x + \psi(x) | x \in \mathcal{B}\} \ltimes (E^1_{1,\dots,k} + E^2_{1,\dots,k} + E^1_{l+1,\dots,n}),$$

where $0 \le k \le l \le n$, $\mathcal{B} \subset \mathfrak{u}(k)$ and $\psi : \mathcal{B} \to E^1_{k+1,\dots,l} + E^2_{k+1,\dots,l} + E^2_{l+1,\dots,n}$ is a surjective linear map with $\psi|_{\mathcal{B}'} = 0$.

Case $m = 0$**.** We have $L = L_0 = \operatorname{span}_{\mathbb{R}}\{e_1, ..., e_n\} = E^1$. If a subgroup $F \subset \operatorname{Sim} \tilde{E}$ preserves L, then F is contained in $(A^1 \times SO(L) \times SO(L^{\perp_\eta})) \leftthreetimes L$. If F acts transitively on L, the we can describe the projection of F to $(A^1 \times SO(L)) \leftthreetimes L$, but the projection of

F to $SO(L) \times SO(L^{\perp_\eta})$ is also contained in $U(n)$. We have

$$(\mathfrak{so}(L) \oplus \mathfrak{so}(L^{\perp_\eta})) \cap \mathfrak{u}(n) = \left\{ \left(\begin{smallmatrix} B & 0 \\ 0 & B \end{smallmatrix}\right) | B \in \mathfrak{so}(L) \right\} = \mathfrak{sod}(1, ..., n).$$

Hence, $(SO(L) \times SO(L^{\perp_\eta})) \cap U(n) = SOD(1, ..., n)$, where $SOD(1, ..., n)$ is the connected Lie subgroup of $U(n)$ corresponding to the Lie algebra $\mathfrak{sod}(1, ..., n)$. Thus the projection of F to $(A^1 \times SO(L)) \rightthreetimes L$ gives us the full information about F.

The Lie algebras corresponding to the transitive similarity transformation groups of type $\mathcal{A}^1$ and φ have the form

$\mathfrak{f}^{0,\mathcal{B},\mathcal{A}^1} = (\mathcal{A}^1 \oplus \mathcal{B}) \ltimes E^1$, where $\mathcal{B} \subset \mathfrak{sod}(1, ..., n)$ is a subalgebra;

$\mathfrak{f}^{0,\mathcal{B},\varphi} = \{x + \varphi(x) | x \in \mathcal{B}\} \ltimes E^1$, where $\varphi : \mathcal{B} \to \mathcal{A}^1$ is a linear map with $\varphi|_{\mathcal{B}'} = 0$.

For a Lie algebra of type ψ we have an orthogonal decomposition $L = W \oplus U$. We choose the vectors $e_1, ..., e_n$ in such a way that $W = \operatorname{span}_{\mathbb{R}}\{e_1, ..., e_k\}$ and $U = \operatorname{span}_{\mathbb{R}}\{e_{k+1}, ..., e_n\}$, where $k = \dim_{\mathbb{R}} W$.

The Lie algebras corresponding to transitively acting groups of type ψ have the form

$$\mathfrak{f}^{0,\mathcal{B},\psi,k} = \{x + \psi(x) | x \in \mathcal{B}\}) \ltimes E^1_{1,\dots,k},$$

where $0 < k < n$, $\mathcal{B} \subset \mathfrak{sod}(1, ..., k)$ and $\psi : \mathcal{B} \to E^1_{k+1,\dots,n}$ is a surjective linear map with $\psi|_{\mathcal{B}'} = 0$.

Case $0 < m < n$**.** In this case $L = \operatorname{span}_{\mathbb{C}}\{e_1, ..., e_m\} \oplus \operatorname{span}_{\mathbb{R}}\{e_{m+1}, ..., e_n\}$. Hence, $L^{\perp_\eta} = \operatorname{span}_{\mathbb{R}}\{f_{m+1}, ..., f_n\}$. Suppose that

$$A = \left(\begin{smallmatrix} B & -C \\ C & B \end{smallmatrix}\right) \in (\mathfrak{so}(L) \oplus \mathfrak{so}(L^{\perp_\eta})) \cap \mathfrak{u}(n),$$

then with respect to the decomposition

$$E = E^1_{1,\dots,m} \oplus E^1_{m+1,\dots,n} \oplus E^2_{1,\dots,m} \oplus E^2_{m+1,\dots,n}$$

the element A has the form $\begin{pmatrix} B_1 & 0 & -C_1 & 0 \\ 0 & B_2 & 0 & 0 \\ C_1 & 0 & B_1 & 0 \\ 0 & 0 & 0 & B_2 \end{pmatrix}$. Consequently, $(\mathfrak{so}(L) \oplus \mathfrak{so}(L^{\perp_\eta})) \cap \mathfrak{u}(n) = \mathfrak{u}(m) \oplus \mathfrak{sod}(m+1, ..., n)$.

Thus, as in case m=0, the projection of an L-preserving subgroup $F \subset \operatorname{Sim} \tilde{E}$ to $\operatorname{Sim} L$ gives us the full information about F.

The Lie algebras corresponding to the transitive similarity transformation groups of type $\mathcal{A}^1$ and φ have the form

$\mathfrak{f}^{m,\mathcal{B},\mathcal{A}^1} = (\mathcal{A}^1 \oplus \mathcal{B}) \ltimes (E^1_{1,...,m} + E^2_{1,...,m} + E^1_{m+1,...,n})$, where $\mathcal{B} \subset \mathfrak{u}(m) \oplus \mathfrak{sod}(m+1,...,n)$ is a subalgebra;

$\mathfrak{f}^{m,\mathcal{B},\varphi} = \{x + \varphi(x) | x \in \mathcal{B}\} \ltimes (E^1_{1,...,m} + E^2_{1,...,m} + E^1_{m+1,...,n})$,
where $\varphi : \mathcal{B} \to \mathcal{A}^1$ is a linear map with $\varphi|_{\mathcal{B}'} = 0$.

Now consider the Lie algebras of type ψ. We have an orthogonal decomposition $L = W \oplus U$. Let $W_1 = W \cap JW$, $U_1 = U \cap JU$, and let W_2 and U_2 be the orthogonal complements to W_1 and U_1 in W and U, respectively. We see that W_1 and U_1 are complex subspaces of $L_0 = L \cap JL = \mathrm{span}_{\mathbb{C}}\{e_1, ..., e_m\}$. For W_2 and U_2 we have $W_2 \cap JW_2 = \{0\}$ and $U_2 \cap JU_2 = \{0\}$. Denote by L_1 the orthogonal complement to L_0 in L and by L_2 the orthogonal complement to $W_1 \oplus U_1$ in L_0. We get the following orthogonal decompositions: $L = L_0 \oplus L_1 = W_1 \oplus U_1 \oplus W_2 \oplus U_2$, $L_0 = W_1 \oplus U_1 \oplus L_2$ and $L_1 \oplus L_2 = W_2 \oplus U_2$. Let $k = \dim_{\mathbb{C}} W_1$, $l = \dim_{\mathbb{C}}(W_1 \oplus U_1)$ and $r = \dim_{\mathbb{R}} W_2 + l$. We see that $0 \le k \le l \le m \le r \le n$. We choose the basis $e_1, ..., e_n, f_1, ..., f_n$ in such a way that

$$W_1 = \mathrm{span}_{\mathbb{C}}\{e_1, ..., e_k\}, U_1 = \mathrm{span}_{\mathbb{C}}\{e_{k+1}, ..., e_l\},$$

$$W_2 = \mathrm{span}_{\mathbb{R}}\{e_{l+1}, ..., e_m\} \oplus \mathrm{span}_{\mathbb{R}}\{e_{m+1}, ..., e_r\}.$$

Then

$$U_2 = \mathrm{span}_{\mathbb{R}}\{f_{l+1}, ..., f_m\} \oplus \mathrm{span}_{\mathbb{R}}\{e_{r+1}, ..., e_n\}.$$

Suppose that $A = \left(\begin{smallmatrix} B & -C \\ C & B \end{smallmatrix}\right) \in (\mathfrak{so}(W) \times \mathfrak{so}(L^{\perp_\eta})) \cap \mathfrak{u}(n)$, then with respect to the decomposition

$$\begin{aligned} E \;=\; & E^1_{1,...,k} \oplus E^1_{k+1,...,l} \oplus E^1_{l+1,...,m} \oplus E^1_{m+1,...,r} \oplus E^1_{r+1,...,n} \oplus \\ & E^2_{1,...,k} \oplus E^2_{k+1,...,l} \oplus E^2_{l+1,...,m} \oplus E^2_{m+1,...,r} \oplus E^2_{r+1,...,n} \end{aligned}$$

the matrix A has the form $\begin{pmatrix} B_1 & 0 & 0 & 0 & 0 & -C_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_1 & 0 & 0 & 0 & 0 & B_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

Hence, $(\mathfrak{so}(W) \times \mathfrak{so}(L^{\perp_\eta})) \cap \mathfrak{u}(n) = \mathfrak{u}(k) \oplus \mathfrak{sod}(m+1, ..., r)$.

Thus the Lie algebras corresponding to transitively acting groups of type ψ have the form

$$\mathfrak{f}^{m,\mathcal{B},\psi,k,l,r} = \{x + \psi(x) | x \in \mathfrak{z}(\mathcal{B})\} \ltimes (E^1_{1,...,k} + E^2_{1,...,k} + E^1_{l+1,...,m} + E^1_{m+1,...,r}),$$

where $0 < k \le l \le m \le r \le n$, $\mathcal{B} \subset \mathfrak{u}(k) \oplus \mathfrak{sod}(m+1,...,r)$ and $\psi : \mathcal{B} \to E^1_{k+1,...,l} + E^2_{k+1,...,l} + E^2_{l+1,...,m} + E^1_{r+1,...,n}$ is a surjective linear map with $\psi|_{\mathcal{B}'} = 0$.

Step 3. Here for each subalgebra $\mathfrak{f} \subset \mathrm{LA}(\mathrm{Sim}\,\tilde{E})$ considered above we describe subalgebras $\mathfrak{a}$ of the Lie algebra $\mathrm{LA}(\mathrm{Sim}\,\mathcal{H}_n)$ with $\pi(\mathfrak{a}) = \mathfrak{f}$.

Case $m > 0$. Let $\mathfrak{f} \subset \mathrm{LA}(\mathrm{Sim}\,\tilde{E})$ be a subalgebra as above with $m > 0$. We claim that if for a subalgebra $\mathfrak{a} \subset \mathrm{LA}(\mathrm{Sim}\,\mathcal{H}_n)$ we have $\pi(\mathfrak{a}) = \mathfrak{f}$, then $\mathfrak{a} = \mathfrak{f} \ltimes \mathcal{C}$, where $\mathcal{C} = \ker \pi$. Indeed, the projection $\pi : \mathrm{LA}(\mathrm{Sim}\,\mathcal{H}_n) \to \mathrm{LA}(\mathrm{Sim}\,\tilde{E})$ is surjective with the kernel $\mathcal{C}$, consequently, if $\pi(\mathfrak{a}) = \mathfrak{f}$ and $\mathfrak{a}$ does not contain $\mathcal{C}$, then $\mathfrak{a}$ has the form $\{x + \zeta(x) | x \in \mathfrak{f}\}$ for some linear map $\zeta : \mathfrak{f} \to \mathcal{C}$. Suppose that $\mathfrak{f} \neq \mathfrak{f}^{m,4,*}$, then we choose $z_1, z_2 \in \mathfrak{f} \cap E$ with $\mathrm{Im}\, g(z_1, z_2) \neq 0$. This yields that $[z_1 + \zeta(z_1), z_2 + \zeta(z_2)] = -2\,\mathrm{Im}\, g(z_1, z_2) \in \mathcal{C}$. If $\mathfrak{f} = \mathfrak{f}^{m,4,*}$, then we can choose $z \in \mathfrak{f} \cap E$ and $x \in \mathfrak{z}(\mathcal{B})$ with $\mathrm{Im}\, g(z, \psi(x)) \neq 0$, or $x_1, x_2 \in \mathfrak{z}(\mathcal{B})$ with $\mathrm{Im}\, g(\psi(x_1), \psi(x_2) \neq 0$. For each subalgebra $\mathfrak{f} \subset \mathrm{LA}(\mathrm{Sim}\,\tilde{E})$ considered above with $m > 0$ we define the Lie subalgebra

$$\mathfrak{a}^{m,*} = \mathfrak{f}^{m,*} \ltimes \mathcal{C} \subset \mathrm{LA}(\mathrm{Sim}\,\mathcal{H}_n).$$

Case m=0. For each subalgebra $\mathfrak{f} \subset \mathrm{LA}(\mathrm{Sim}\,\tilde{E})$ considered above with $m = 0$ we have the following two possibilities:

Subcase 1. Here we have $\mathfrak{a}^{0,*} = \mathfrak{f}^{0,*} \ltimes \mathcal{C}$.

Subcase 2. The Lie algebra $\mathfrak{a}$ does not contain $\mathcal{C}$. Hence $\mathfrak{a}$ has the form $\{x + \zeta(x) | x \in \mathfrak{f}\}$ for some linear map $\zeta : \mathfrak{f} \to \mathcal{C}$. For each $\mathfrak{f}^{0,*}$ we will find all possible ζ. Since $\mathfrak{a}$ is a Lie algebra, we see that ζ vanishes on the commutator $\mathfrak{f}'$.

Subcase 2.1. Consider $\mathfrak{f}^{0,\mathcal{B},\mathcal{A}^1} = (\mathcal{A}^1 \oplus \mathcal{B}) \ltimes E^1$, where $\mathcal{B} \subset \mathfrak{sod}(1,...,n)$ is a subalgebra. Since $(\mathfrak{f}^{0,1,\mathcal{B}})' = E^1 + \mathcal{B}'$, we obtain $\zeta|_{E^1 + \mathcal{B}'} = 0$. Let $a \in \mathcal{A}^1$ and $x \in \mathfrak{z}(\mathcal{B})$. From $[a + \zeta(a), x + \zeta(x)] = a\zeta(x) \in \mathcal{C}$, it follows that $\zeta|_{\mathfrak{z}(\mathcal{B})} = 0$. Thus ζ can be considered as a linear map $\zeta : \mathcal{A}^1 \to \mathcal{C}$. For any linear map $\zeta : \mathcal{A}^1 \to \mathcal{C}$ we consider the Lie algebra

$$\mathfrak{a}^{0,\mathcal{B},\mathcal{A}^1,\zeta} = (\mathcal{B} \oplus \{a + \zeta(a) | a \in \mathcal{A}^1\}) \ltimes E^1.$$

Subcase 2.2. Consider $\mathfrak{f}^{0,\mathcal{B},\varphi}$ with $\varphi = 0$, i.e. $\mathfrak{f}^{0,\mathcal{B},0} = \mathcal{B} \ltimes E^1$, where $\mathcal{B} \subset \mathfrak{sod}(1,...,n)$ is a subalgebra. We have $(\mathfrak{f}^{0,2,\mathcal{B}})' = \mathcal{B}' + \mathrm{span}\{x(u) | x \in \mathcal{B}, u \in E^1\}$. Choose the vectors $e_1, ..., e_n$ so that $E^1_{i_0+1,...,n}$ is the subspace of E^1 annihilated by $\mathcal{B}$ and $E^1_{1,...,i_0}$ is the orthogonal complement to $E^1_{i_0+1,...,n}$ in E^1. The Lie algebra $\mathcal{B}$ is compact, hence $\mathcal{B}$ is totally reducible and $E^1_{1,...,i_0}$ is decomposed into an orthogonal sum of subspaces, on each of these subspaces $\mathcal{B}$ acts irreducibly. Thus, $\mathrm{span}\{x(u) | x \in \mathcal{B}, u \in E^1\} = E^1_{1,...,i_0}$, and ζ can be con-

sidered as a linear map $\zeta : \mathfrak{z}(\mathcal{B}) \oplus E^1_{i_0+1,\dots,n} \to \mathcal{C}$. For any linear map $\zeta : \mathcal{B} \oplus E^1_{i_0+1,\dots,n} \to \mathcal{C}$ with $\zeta|_{\mathcal{B}'} = 0$ we define the Lie algebra

$$\mathfrak{a}^{0,\mathcal{B},\varphi=0,i_0,\zeta} = \{x + \zeta(x) | x \in \mathcal{B} \oplus E^1_{i_0+1,\dots,n}\} \ltimes E^1_{1,\dots,i_0}.$$

Subcase 2.3. Consider $\mathfrak{f}^{0,\mathcal{B},\varphi} = \{x + \varphi(x) | x \in \mathcal{B}\}) \ltimes E^1$, where $\mathcal{B} \subset \mathfrak{sod}(1,...,n)$ and $\varphi : \mathcal{B} \to \mathcal{A}^1$ is a non-zero linear map with $\varphi|_{\mathcal{B}'} = 0$. As above, we can show that $(\mathfrak{f}^{0,\mathcal{B},\varphi})' = E^1 + \mathcal{B}'$. Hence ζ is a map $\zeta : \{x + \varphi(x) | x \in \mathfrak{z}(\mathcal{B})\} \to \mathcal{C}$. Denote by the same letter ζ the linear map $\zeta : \mathfrak{z}(\mathcal{B}) \to \mathcal{C}$ defined by $\zeta(x) = \zeta(x + \varphi(x))$. Let $x_1, x_2 \in \mathfrak{z}(\mathcal{B})$ we have $[x_1 + \varphi(x_1) + \zeta(x_1), x_2 + \varphi(x_2) + \zeta(x_2)] = \varphi(x_1)\zeta(x_2) - \varphi(x_2)\zeta(x_1) \in \mathcal{C}$. Hence, $\varphi(x_1)\zeta(x_2) = \varphi(x_2)\zeta(x_1)$. In particular, if $\varphi(x) = 0$, then $\zeta(x) = 0$. Hence, $\ker \varphi \subset \ker \zeta$. Conversely, if $\ker \varphi \subset \ker \zeta$, then $\varphi(x_1)\zeta(x_2) = \varphi(x_2)\zeta(x_1)$ for all $x_1, x_2 \in \mathfrak{z}(\mathcal{B})$.

Let i_0 be as in Subcase 2.2. For any linear map $\zeta : \mathcal{B} \oplus E^1_{i_0+1,\dots,n} \to \mathcal{C}$ such that $\ker \varphi \subset \ker \zeta$ and $\zeta|_{E^1_{i_0+1,\dots,n}} = 0$ we consider the Lie algebra

$$\mathfrak{a}^{0,\mathcal{B},\varphi,i_0,\zeta} = \{x + \varphi(x) + \zeta(x) | x \in \mathcal{B} \oplus E^1_{i_0+1,\dots,n}\} \ltimes E^1_{1,\dots,i_0}.$$

Note that the Lie algebras $\mathfrak{a}^{0,\mathcal{B},\varphi,i_0,\zeta}$ for $\varphi = 0$ and $\varphi \neq 0$ are defined in the same way.

Subcase 2.4. As above, for the Lie algebra

$$\mathfrak{f}^{0,\mathcal{B},\psi,k} = \{x + \psi(x) | x \in \mathcal{B}\} \ltimes E^1_{1,\dots,k},$$

where $0 < k < n$, $\mathcal{B} \subset \mathfrak{sod}(1,...,k)$ and $\psi : \mathcal{B} \to E^1_{k+1,\dots,n}$ is a surjective linear map with $\psi|_{\mathcal{B}'} = 0$, we can prove that the map ζ vanishes on $\mathcal{B}' + E^1_{1,\dots,i_0}$, where i_0 is as in Subcase 2.2. For any linear map $\zeta : \mathcal{B} \oplus E^1_{i_0+1,\dots,k} \to \mathcal{C}$ with $\zeta|_{\mathcal{B}'} = 0$ we consider the Lie algebra

$$\mathfrak{f}^{0,\mathcal{B},\psi,,k,i_0,\zeta} = \{x + \psi(x) + \zeta(x) | x \in \mathcal{B}\} \oplus \{u + \zeta(u) | u \in E^1_{i_0+1,\dots,k}\}) \ltimes E^1_{1,\dots,i_0}.$$

Step 4. For each subalgebra $\mathfrak{a} \subset \mathrm{LA}(\mathrm{Sim}\,\mathcal{H}_n)$ constructed above consider the subalgebra $\Gamma_0^{-1}(\mathfrak{a}) \subset \mathfrak{su}(1,n+1)_{<p_1,p_2>}$. Note that we have $\Gamma_0(\mathcal{N}^1_{k,\dots,l}) = E^1_{k,\dots,l}$, $\Gamma_0(\mathcal{N}^2_{k,\dots,l}) = E^2_{k,\dots,l}$, $\Gamma_0(\mathcal{A}^1) = \mathcal{A}^1$ and $\Gamma_0(\mathcal{C}) = \mathcal{C}$, where we consider $\mathcal{A}^1$ and $\mathcal{C}$ as the subalgebras of $\mathfrak{su}(1,n+1)_{<p_1,p_2>}$ as well as of $\mathrm{LA}(\mathrm{Sim}\,\mathcal{H}_n)$. Let $\mathfrak{a}^{m,\mathcal{B}} \subset \mathrm{LA}(\mathrm{Sim}\,\mathcal{H}_n)$ be any subalgebra constructed above with the associated number $0 \leq m \leq n$ and the associated subalgebra $\mathcal{B} \subset \mathfrak{u}(m) \oplus \mathfrak{sod}(m+1,..,n) = \mathfrak{su}(m) \oplus \mathbb{R}J_m \oplus \mathfrak{sod}(m+1,...,n)$. We have $\Gamma|_{\mathfrak{su}(n)} = \mathrm{id}_{\mathfrak{su}(n)}$, where $\mathfrak{su}(n)$ is considered as the subalgebras of $\mathfrak{su}(1,n+1)_{<p_1,p_2>}$ and of $\mathrm{LA}(\mathrm{Sim}\,\mathcal{H}_n)$. Furthermore, since $J_m - \frac{m}{n}J_n \in \mathfrak{su}(n)$, we have $\Gamma_0^{-1}(J_m) = \Gamma_0^{-1}(J_m - \frac{m}{n}J_n) + \frac{m}{n}\Gamma_0^{-1}(J_n) = J_m - \frac{m}{n}J_n +$

$\frac{m}{n}I_0 = J_m - \frac{m}{n}J_n + \frac{m}{n}(0, -\frac{n}{n+2}, 0, \frac{2}{n+2}E_n, 0, 0, 0) = J_m - \frac{m}{n+2}J$. Let $\mathfrak{h} = \mathrm{pr}_{\mathfrak{u}(n)}\Gamma_0^{-1}(\mathfrak{a}) = \mathrm{pr}_{\mathfrak{u}(n)}\Gamma_0^{-1}(\mathcal{B}) \subset \mathfrak{su}(m) \oplus \mathbb{R}(J_m - \frac{m}{n+2}J_n)$. Thus,

$$\Gamma_0^{-1}(\mathcal{B}) = \{(0, -\frac{1}{2}\operatorname{tr} C, B, C, 0, 0, 0) | \left(\begin{smallmatrix} B & -C \\ C & B \end{smallmatrix}\right) \in \mathfrak{h}\}.$$

For the Lie algebra $\mathfrak{a}^{m,\mathfrak{h},\psi,k,l,r}$ with $0 < m < n$ we have $\mathcal{B} \subset \mathfrak{su}(k) \oplus \mathbb{R}J_k \oplus \mathfrak{sod}(m+1, ..., r)$, consequently, $\mathfrak{h} \subset \mathfrak{su}(k) \oplus \mathbb{R}(J_k - \frac{k}{n+2}J_n) \oplus \mathfrak{sod}(m+1, ..., r)$. For the Lie algebra $\mathfrak{a}^{n,\mathfrak{h},\psi,k,l}$ we have $\mathcal{B} \subset \mathfrak{su}(k) \oplus \mathbb{R}J_k$, hence, $\mathfrak{h} \subset \mathfrak{su}(k) \oplus \mathbb{R}(J_k - \frac{k}{n+2}J_n)$. For the Lie algebra $\mathfrak{a}^{0,\mathfrak{h},\psi,k}$ we have $\mathcal{B} \subset \mathfrak{sod}(1, ..., k)$, hence $\mathfrak{h} \subset \mathfrak{sod}(1, ..., k)$. We denote $\Gamma_0^{-1}(\mathfrak{a}^{m,\mathcal{B},*})$ by $\mathfrak{g}^{m,\mathcal{B},*}$.

Thus we obtain a list of subalgebras of $\mathfrak{su}(1, n+1)_{<p_1,p_2>}$. From proposition 2.1 and Theorem 2.5 it follows that this list contains all weakly-irreducible subalgebras of $\mathfrak{su}(1, n+1)_{<p_1,p_2>}$. Example 2.3 shows that this list contains also subalgebras of $\mathfrak{su}(1, n+1)_{<p_1,p_2>}$ that are not weakly-irreducible. Here we verify which of the subalgebras $\Gamma_0^{-1}(\mathfrak{a}) \subset \mathfrak{su}(1, n+1)_{<p_1,p_2>}$ are weakly-irreducible.

The subalgebras of the form $\mathfrak{g}^{m,\mathfrak{h},\mathcal{A}^1}$ and $\mathfrak{g}^{m,\mathfrak{h},\varphi}$, where $0 \leq m \leq n$ contain $\mathcal{N}^1 + \mathcal{C}$, hence these subalgebras are weakly-irreducible.

Lemma 2.1. *The subalgebras of the form* $\mathfrak{g}^{0,\mathfrak{h},\psi,k}$, $\mathfrak{g}^{m,\mathfrak{h},\psi,k,l,r}$ *and* $\mathfrak{g}^{n,\mathfrak{h},\psi,k,l}$ *are weakly-irreducible.*

Proof. Let $\mathfrak{g}$ be any of these subalgebras and suppose $\mathfrak{g}$ preserves a proper vector subspace $L \subset \mathbb{R}^{2,2n+2}$. As in Example 2.1, we can show that if $(a_1, a_2, \alpha, \beta, b_1, b_2) \in L$, then we have $-b_2p_1 + b_1p_2 \in L$ and $b_1p_1 + b_2p_2 \in L$. We suppose that $L \subset \mathbb{R}p_1 \oplus \mathbb{R}p_2 \oplus E$. Let $v = (a_1, a_2, \alpha, \beta, 0, 0) \in L$. We assume $\alpha \neq 0$. If the projection of α to W is non-trivial, then we denote this projection by w. Applying the element $(0, 0, 0, 0, w, 0, 0) \in \mathfrak{g}$ to v, we obtain $w^t\alpha p_1 + w^t\beta p_2 \in L$. Consequently L is degenerate subspace. Suppose that the projections of α and β to W are trivial and the projection of α to U is non-trivial, then there exists an element $x \in \mathfrak{z}(\mathfrak{h})$ such that $\psi(x) = (0, 0, 0, 0, u, 0, 0)$, where $u \in U$ is equal to this projection. The element $x \in \mathfrak{z}(\mathfrak{h})$ has the form $x_1 + \mu(J_k - \frac{k}{n+2}J_n)$, where $x_1 \in \mathfrak{su}(k) \oplus \mathfrak{sod}(m+1, ..., r)$ (resp. $x_1 \in \mathfrak{su}(k)$ and $x_1 \in \mathfrak{sod}(1, ..., k)$) for the Lie algebra $\mathfrak{g}^{m,\mathfrak{h},\psi,k,l,r}$ (resp. $g^{0,\mathfrak{h},\psi,k}$ and $\mathfrak{g}^{n,\mathfrak{h},\psi,k,l}$), $\mu = 0$ or 1 for the Lie algebras $\mathfrak{g}^{m,\mathfrak{h},\psi,k,l,r}$ and $\mathfrak{g}^{n,\mathfrak{h},\psi,k,l}$, and $\mu = 0$ for the Lie algebra $\mathfrak{g}^{0,\mathfrak{h},\psi,k}$. Consider the element $X = (0, -\frac{\mu k}{n+2}, B_1, C_1 + \mu(J_k - \frac{k}{n+2}J_n), u, 0, 0, 0) \in \mathfrak{g}$, where $\left(\begin{smallmatrix} B_1 & -C_1 \\ C_1 & B_1 \end{smallmatrix}\right) = x_1$. Applying the element X to v we get

$Xv = (\frac{\mu k}{n+2}a_2 + u^t\alpha, -\frac{\mu k}{n+2}a_2 + u^t\beta, \frac{\mu k}{n+2}\beta, -\frac{\mu k}{n+2}\alpha, 0, 0) \in L, \qquad X^2v =$
$(\frac{\mu k}{n+2}(-\frac{\mu k}{n+2}a_1 + u^t\beta) + \frac{\mu k}{n+2}u^t\beta, -\frac{\mu k}{n+2}(\frac{\mu k}{n+2}a_2 + u^t\alpha) - \frac{\mu k}{n+2}u^t\alpha, -(\frac{\mu k}{n+2})^2\alpha, -(\frac{\mu k}{n+2})^2\beta, 0, 0) \in$

L. Hence if $\mu = 1$, then $v + (\frac{\mu k}{n+2})^2 X^2 v = (2\frac{\mu k}{n+2} u^t \beta, 2\frac{\mu k}{n+2} u^t \alpha, 0, 0, 0, 0) \in L$. If $\mu = 0$, then $Xv = (u^t \alpha, u^t \beta, 0, 0, 0, 0) \in L$.

Thus the subspace $L \subset \mathbb{R}^{2,2n+2}$ is degenerate and the lemma is proved. $\square$

We are left now with the Lie algebras of the form $\mathfrak{g}^{0,\mathfrak{h},\mathcal{A}^1,\zeta}$, $\mathfrak{g}^{0,\mathfrak{h},\varphi,i_0,\zeta}$ and $\mathfrak{g}^{0,\mathfrak{h},\psi,k,i_0,\zeta}$.

Lemma 2.2. *The subalgebra* $\mathfrak{g}^{0,\mathfrak{h},\mathcal{A}^1,\zeta} \subset \mathfrak{su}(1, n+1)_{<p_1,p_2>}$ *is not weakly-irreducible.*

Proof. The Lie algebra $\mathfrak{g}^{0,\mathfrak{h},\mathcal{A}^1,\zeta}$ preserves the non-degenerate proper vector subspace $\mathrm{span}_{\mathbb{R}}\{p_1 + p_2, e_1 + f_1, ..., e_n + f_n, q_1 - \frac{\zeta}{2} p_2 + q_2 + \frac{\zeta}{2} p_1\} \subset \mathbb{R}^{2,2n+2}$. $\square$

Lemma 2.3. *The subalgebra* $\mathfrak{g}^{0,\mathfrak{h},\varphi,i_0,\zeta} \subset \mathfrak{su}(1, n+1)_{<p_1,p_2>}$ *is weakly-irreducible if and only if* $\zeta|_{\mathfrak{z}(\mathfrak{h})} \neq 0$ *and* $\varphi = 0$.

Proof. If $\varphi \neq 0$, then $\mathfrak{g}^{0,\mathfrak{h},\varphi,i_0,\zeta}$ preserves the non-degenerate proper vector subspace $\mathrm{span}_{\mathbb{R}}\{p_1 + p_2, e_1 + f_1, ..., e_n + f_n, q_1 - \frac{\zeta(A)}{2\varphi(A)} p_2 + q_2 + \frac{\zeta(A)}{2\varphi(A)} p_1\} \subset \mathbb{R}^{2,2n+2}$, where $A \in \mathfrak{z}(\mathfrak{h})$ is a non-zero element that is orthogonal to $\ker \varphi$.

Now assume that $\varphi = 0$. Suppose that $\zeta|_{\mathfrak{z}(\mathfrak{h})} = 0$. If $\zeta = 0$, then $\mathfrak{g}^{0,\mathfrak{h},\varphi,i_0,\zeta}$ preserves the non-degenerate vector subspaces $\mathrm{span}_{\mathbb{R}}\{p_1 + p_2, e_1 + f_1, ..., e_n + f_n, q_1 + q_2\} \subset \mathbb{R}^{2,2n+2}$ and $\mathrm{span}_{\mathbb{R}}\{p_1 - p_2, e_1 - f_1, ..., e_n - f_n, q_1 - q_2\} \subset \mathbb{R}^{2,2n+2}$. If $\zeta \neq 0$, then we choose the vectors $e_{i_0+1}, ..., e_n$ so that $\zeta|_{\mathrm{span}\{e_{i_0+1},...,e_{n-1}\}} = 0$. Then $\mathfrak{g}^{0,2,\mathfrak{h},i_0,\zeta}$ preserves the non-degenerate vector subspace $\mathrm{span}_{\mathbb{R}}\{p_1 + p_2, e_1 + f_1, ..., e_n + f_n, q_1 + q_2 - 2\zeta(e_n)e_n\} \subset \mathbb{R}^{2,2n+2}$.

Suppose that $\zeta|_{\mathfrak{z}(\mathfrak{h})} \neq 0$. Let $x = \left(\begin{smallmatrix} B & 0 \\ 0 & B \end{smallmatrix}\right) \in \mathfrak{z}(\mathfrak{h})$ with $\zeta(x) \neq 0$. Since $B \in \mathfrak{so}(n)$, we can choose the basis $e_1, ..., e_n$ so that B has the form $\begin{pmatrix} \begin{smallmatrix} 0 & -\lambda_1 \\ \lambda_1 & 0 \end{smallmatrix} & & & & 0 \\ & \ddots & & & \\ & & \begin{smallmatrix} 0 & -\lambda_s \\ \lambda_s & 0 \end{smallmatrix} & & \\ & & & 0 & \\ 0 & & & & \ddots \; 0 \end{pmatrix}$, where $2s \leq i_0$ and $\lambda_i \neq 0$.

Suppose that $\mathfrak{g}^{0,2,\mathfrak{h},i_0,\zeta}$ preserves a proper vector subspace $L \subset \mathbb{R}^{2,2n+2}$. Let $v = (a_1, a_2, \alpha, \beta, b_1, b_2) \in L$ and $X = x + \zeta(x) \in \mathfrak{g}^{0,\mathfrak{h},\varphi,i_0,\zeta}$. We have

$$Xv = \begin{pmatrix} -\zeta(x)b_2 \\ \zeta(x)b_1 \\ -\lambda_1\alpha_2 \\ \lambda_1\alpha_1 \\ \vdots \\ -\lambda_s\alpha_s \\ \lambda_s\alpha_{s-1} \\ 0 \\ \vdots \\ 0 \\ -\lambda_1\beta_2 \\ \lambda_1\beta_1 \\ \vdots \\ -\lambda_s\beta_s \\ \lambda_s\beta_{s-1} \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix} \in L, \quad X^3v = -\begin{pmatrix} 0 \\ 0 \\ \lambda_1^2(-\lambda_1\alpha_2) \\ \lambda_1^2(\lambda_1\alpha_1) \\ \vdots \\ \lambda_s^2(-\lambda_s\alpha_s) \\ \lambda_s^2(\lambda_s\alpha_{s-1}) \\ 0 \\ \vdots \\ 0 \\ \lambda_1^2(-\lambda_1\beta_2) \\ \lambda_1^2(\lambda_1\beta_1) \\ \vdots \\ \lambda_s^{2s}(-\lambda_s\beta_s) \\ \lambda_s^2(\lambda_s\beta_{s-1}) \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix} \in L, \quad \cdots, \quad X^{2s+1}v = -\begin{pmatrix} 0 \\ 0 \\ \lambda_1^{2s}(-\lambda_1\alpha_2) \\ \lambda_1^{2s}(\lambda_1\alpha_1) \\ \vdots \\ \lambda_s^{2s}(-\lambda_s\alpha_s) \\ \lambda_s^{2s}(\lambda_s\alpha_{s-1}) \\ 0 \\ \vdots \\ 0 \\ \lambda_1^{2s}(-\lambda_1\beta_2) \\ \lambda_1^{2s}(\lambda_1\beta_1) \\ \vdots \\ \lambda_s^{2s}(-\lambda_s\beta_s) \\ \lambda_s^{2s}(\lambda_s\beta_{s-1}) \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix} \in L.$$

The vector $-\zeta(x)b_2p_1 + \zeta(x)b_1p_2$ can by decomposed into a combination of the vectors Xv, X^3v,...,$X^{2s+1}v \in L$, hence $-b_2p_1 + b_1p_2 \in L$. The end of the proof of the lemma is as in Example 2.1. $\square$

Lemma 2.4. *The subalgebra $\mathfrak{g}^{0,\mathfrak{h},\psi,k,i_0,\zeta} \subset \mathfrak{su}(1,n+1)_{<p_1,p_2>}$ is weakly-irreducible if and only if $\zeta|_{\mathfrak{z}(\mathfrak{h})} \neq 0$.*

The proof is similar to the proofs of Lemma 2.1 and 2.3. $\square$

Now for the Lie algebra $\mathfrak{g}^{0,\mathfrak{h},\psi,k,i_0,\zeta}$ we assume that $\zeta|_{\mathfrak{z}(\mathfrak{h})} \neq 0$. Obviously, this Lie algebra has the form $\mathfrak{g}^{0,\mathfrak{h},\psi',k',i_0,\zeta'}$, where $\zeta'|_{E^1_{i_0+1,\dots,k'}} = 0$. We will denote this Lie algebra by $\mathfrak{g}^{0,\mathfrak{h},\psi',k,\zeta'}$. Consider the Lie algebra $\mathfrak{g}^{0,\mathfrak{h},\varphi,i_0,\zeta}$ such that $\zeta|_{E^1_{i_0+1,\dots,n}} \neq 0$, $\zeta|_{\mathfrak{z}(\mathfrak{h})} \neq 0$ and $\varphi = 0$. Obviously, this is the Lie algebra of the form $\mathfrak{g}^{0,\mathfrak{h},\psi,k,\zeta}$. Thus we may restrict our attention to the Lie algebras $\mathfrak{g}^{0,\mathfrak{h},\varphi,i_0,\zeta}$ such that $\zeta|_{E^1_{i_0+1,\dots,n}} = 0$, $\zeta|_{\mathfrak{z}(\mathfrak{h})} \neq 0$ and $\varphi = 0$. We denote such Lie algebra by $\mathfrak{g}^{0,\mathfrak{h},\zeta}$.

The theorem is proved. $\square$

3 Classification of the holonomy algebras and constructions of metrics

In this part we give the classification of the holonomy algebras of pseudo-Kählerian manifolds of index 2. For each weakly-irreducible not irreducible holonomy algebra $\mathfrak{g}$ we construct a polynomial metric with the holonomy algebra $\mathfrak{g}$. The results are stated in Section 3.1. In the other two sections we give the proofs of the theorems.

3.1 Main results

In the following theorem we give the classification of the weakly-irreducible not irreducible holonomy algebras of pseudo-Kählerian manifolds of index 2. We use the denotation from Section 2.1.

Theorem 3.1. 1) *A subalgebra $\mathfrak{g} \subset \mathfrak{u}(1,1)$ is the weakly-irreducible not irreducible holonomy algebra of a pseudo-Kählerian manifold of signature $(2,2)$ if and only if $\mathfrak{g}$ is conjugated to one of the following subalgebras of $\mathfrak{u}(1,1)_{<p_1,p_2>}$:*

$\mathfrak{hol}^1_{n=0} = \mathfrak{u}(1,1)_{<p_1,p_2>}$;

$$\mathfrak{hol}^2_{n=0} = \mathcal{A}^1 \oplus \mathcal{A}^2 = \left\{ \left(\begin{array}{cccc} a_1 & -a_2 & 0 & 0 \\ a_2 & a_1 & 0 & 0 \\ 0 & 0 & -a_1 & -a_2 \\ 0 & 0 & a_2 & -a_1 \end{array} \right) \middle| \, a_1, a_2 \in \mathbb{R} \right\};$$

$$\mathfrak{hol}^{\gamma_1,\gamma_2}_{n=0} = \{(a\gamma_1, a\gamma_2, 0) | a \in \mathbb{R}\} \ltimes \mathcal{C} = \left\{ \left(\begin{array}{cccc} a\gamma_1 & -a\gamma_2 & 0 & -c \\ a\gamma_2 & a\gamma_1 & c & 0 \\ 0 & 0 & -a\gamma_1 & -a\gamma_2 \\ 0 & 0 & a\gamma_2 & -a\gamma_1 \end{array} \right) \middle| \, a, c \in \mathbb{R} \right\}, \text{ where } \gamma_1, \gamma_2 \in \mathbb{R}.$$

2) *Let $n \geq 1$. Then a subalgebra $\mathfrak{g} \subset \mathfrak{u}(1, n+1)$ is the weakly-irreducible not irreducible holonomy algebra of a pseudo-Kählerian manifold of signature $(2, 2n+2)$ if and only if $\mathfrak{g}$ is conjugated to one of the following subalgebras of $\mathfrak{u}(1, n+1)_{<p_1,p_2>}$:*

$$\mathfrak{hol}^{m,\mathfrak{u},\mathcal{A}^1,\tilde{\mathcal{A}}^2} = (\mathcal{A}^1 \oplus \tilde{\mathcal{A}}^2 \oplus \mathfrak{u}) \ltimes (\mathcal{N}^1 + \mathcal{N}^2_{1,\dots,m} + \mathcal{C})$$

$$= \left\{ \left(\begin{array}{cccccccc} a_1 & -a_2 & -z_1^t & -z_1'^t & -z_2^t & 0 & 0 & -c \\ a_2 & a_1 & z_2^t & 0 & -z_1^t & -z_1'^t & c & 0 \\ 0 & 0 & B & 0 & -C & 0 & z_1 & -z_2 \\ 0 & 0 & 0 & 0 & 0 & -a_2 E_{n-m} & z_1' & 0 \\ 0 & 0 & C & 0 & B & 0 & z_2 & z_1 \\ 0 & 0 & 0 & a_2 E_{n-m} & 0 & 0 & 0 & z_1' \\ 0 & 0 & 0 & 0 & 0 & 0 & -a_1 & -a_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_2 & -a_1 \end{array} \right) \middle| \, \begin{array}{c} a_1, a_2, c \in \mathbb{R}, \\ z_1, z_2 \in \mathbb{R}^m, \\ z_1' \in \mathbb{R}^{n-m}, \\ \left(\begin{smallmatrix} B & -C \\ C & B \end{smallmatrix}\right) \in \mathfrak{u} \end{array} \right\},$$

where $0 \le m \le n$ and $\mathfrak{u} \subset \mathfrak{u}(m)$ is a subalgebra;

$\mathfrak{hol}^{m,\mathfrak{u},\mathcal{A}^1,\phi} = \{(a_1, \phi(B,C), B, C, 0, 0, 0) + \phi(B,C)J_{m+1,\dots,n} | a_1 \in \mathbb{R}, \left(\begin{smallmatrix} B & -C \\ C & B \end{smallmatrix}\right) \in \mathfrak{u}\} \ltimes (\mathcal{N}^1 + \mathcal{N}^2_{m+1,\dots,n} + \mathcal{C})$

$$= \left\{ \left(\begin{array}{cccccccc} a_1 & -\phi(A) & -z_1^t & -z_1'^t & -z_2^t & 0 & 0 & -c \\ \phi(A) & a_1 & z_2^t & 0 & -z_1^t & -z_1'^t & c & 0 \\ 0 & 0 & B & 0 & -C & 0 & z_1 & -z_2 \\ 0 & 0 & 0 & 0 & 0 & -\phi(A)E_{n-m} & z_1' & 0 \\ 0 & 0 & C & 0 & B & 0 & z_2 & z_1 \\ 0 & 0 & 0 & \phi(A)E_{n-m} & 0 & 0 & 0 & z_1' \\ 0 & 0 & 0 & 0 & 0 & 0 & -a_1 & -\phi(A) \\ 0 & 0 & 0 & 0 & 0 & 0 & \phi(A) & -a_1 \end{array} \right) \middle| \begin{array}{c} a_1, c \in \mathbb{R}, \\ z_1, z_2 \in \mathbb{R}^m, \\ z_1' \in \mathbb{R}^{n-m}, \\ A = \\ \left(\begin{smallmatrix} B & -C \\ C & B \end{smallmatrix}\right) \in \mathfrak{u} \end{array} \right\},$$

where $0 \le m \le n$, $\mathfrak{u} \subset \mathfrak{u}(m)$ is a subalgebra and $\phi : \mathfrak{u} \to \mathbb{R}$ is a linear map with $\phi|_{\mathfrak{u}'} = 0$;

$\mathfrak{hol}^{m,\mathfrak{u},\varphi,\phi} = \{(\varphi(B,C), \phi(B,C), B, C, 0, 0, 0) + \phi(B,C)J_{m+1,\dots,n} | \left(\begin{smallmatrix} B & -C \\ C & B \end{smallmatrix}\right) \in \mathfrak{u}\} \ltimes (\mathcal{N}^1 + \mathcal{N}^2_{m+1,\dots,n} + \mathcal{C})$

$$= \left\{ \left(\begin{array}{cccccccc} \varphi(A) & -\phi(A) & -z_1^t & -z_1'^t & -z_2^t & 0 & 0 & -c \\ \phi(A) & \varphi(A) & z_2^t & 0 & -z_1^t & -z_1'^t & c & 0 \\ 0 & 0 & B & 0 & -C & 0 & z_1 & -z_2 \\ 0 & 0 & 0 & 0 & 0 & -\phi(A)E_{n-m} & z_1' & 0 \\ 0 & 0 & C & 0 & B & 0 & z_2 & z_1 \\ 0 & 0 & 0 & \phi(A)E_{n-m} & 0 & 0 & 0 & z_1' \\ 0 & 0 & 0 & 0 & 0 & 0 & -\varphi(A) & -\phi(A) \\ 0 & 0 & 0 & 0 & 0 & 0 & \phi(A) & -\varphi(A) \end{array} \right) \middle| \begin{array}{c} c \in \mathbb{R}, \\ z_1, z_2 \in \mathbb{R}^m, \\ z_1' \in \mathbb{R}^{n-m}, \\ A = \\ \left(\begin{smallmatrix} B & -C \\ C & B \end{smallmatrix}\right) \in \mathfrak{u} \end{array} \right\},$$

where $0 \le m \le n$, $\mathfrak{u} \subset \mathfrak{u}(m)$ is a subalgebra and $\varphi, \phi : \mathfrak{u} \to \mathbb{R}$ are linear maps with $\varphi|_{\mathfrak{u}'} = \phi|_{\mathfrak{u}'} = 0$;

$\mathfrak{hol}^{m,\mathfrak{u},\varphi\tilde{\mathcal{A}}^2} = \{(\varphi(B,C), a_2, B, C, 0, 0, 0) + a_2 J_{m+1,\dots,n} | a_2 \in \mathbb{R}, \left(\begin{smallmatrix} B & -C \\ C & B \end{smallmatrix}\right) \in \mathfrak{u}\} \ltimes (\mathcal{N}^1 + \mathcal{N}^2_{m+1,\dots,n} + \mathcal{C})$

$$= \left\{ \left(\begin{array}{cccccccc} \varphi(A) & -a_2 & -z_1^t & -z_1'^t & -z_2^t & 0 & 0 & -c \\ a_2 & \varphi(A) & z_2^t & 0 & -z_1^t & -z_1'^t & c & 0 \\ 0 & 0 & B & 0 & -C & 0 & z_1 & -z_2 \\ 0 & 0 & 0 & 0 & 0 & -a_2E_{n-m} & z_1' & 0 \\ 0 & 0 & C & 0 & B & 0 & z_2 & z_1 \\ 0 & 0 & 0 & a_2E_{n-m} & 0 & 0 & 0 & z_1' \\ 0 & 0 & 0 & 0 & 0 & 0 & -\varphi(A) & -a_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_2 & -\varphi(A) \end{array} \right) \middle| \begin{array}{c} a_2, c \in \mathbb{R}, \\ z_1, z_2 \in \mathbb{R}^m, \\ z_1' \in \mathbb{R}^{n-m}, \\ A = \\ \left(\begin{smallmatrix} B & -C \\ C & B \end{smallmatrix}\right) \in \mathfrak{u} \end{array} \right\},$$

where $0 \le m \le n$, $\mathfrak{u} \subset \mathfrak{u}(m)$ is a subalgebra and $\varphi : \mathfrak{u} \to \mathbb{R}$ is a linear map with $\varphi|_{\mathfrak{u}'} = 0$;

$$\mathfrak{hol}^{m,\mathfrak{u},\lambda} = (\{(a_1,\lambda a_1,0,0,0,0,0)+\lambda a_1 J_{m+1,\dots,n}|a_1\in\mathbb{R}\}\oplus\mathfrak{u})\ltimes(\mathcal{N}^1+\mathcal{N}^2_{m+1,\dots,n}+\mathcal{C})$$

$$= \left\{\left.\begin{pmatrix} a_1 & -\lambda a_1 & -z_1^t & -z_1'^t & -z_2^t & 0 & 0 & -c\\ \lambda a_1 & a_1 & z_2^t & 0 & -z_1^t & -z_1'^t & c & 0\\ 0 & 0 & B & 0 & -C & 0 & z_1 & -z_2\\ 0&0&0&0&0&-\lambda a_1 E_{n-m} & z_1' & 0\\ 0&0&C&0&B&0&z_2&z_1\\ 0&0&0&\lambda a_1 E_{n-m}&0&0&0&z_1'\\ 0&0&0&0&0&0&-a_1&-\lambda a_1\\ 0&0&0&0&0&0&\lambda a_1&-a_1 \end{pmatrix}\right| \begin{array}{c} a_1,c\in\mathbb{R},\\ z_1,z_2\in\mathbb{R}^m,\\ z_1'\in\mathbb{R}^{n-m},\\ \left(\begin{smallmatrix} B & -C\\ C & B\end{smallmatrix}\right)\in\mathfrak{u}\end{array}\right\},$$

where $0\le m\le n$, $\mathfrak{u}\subset\mathfrak{u}(m)$ is a subalgebra and $\lambda\in\mathbb{R}$, $\lambda\neq 0$;

$$\begin{aligned}\mathfrak{hol}^{n,\mathfrak{u},\psi,k,l} = {} & \{(0,0,B,C,\psi_1(B,C),\psi_2(B,C)+\psi_3(B,C),0)|\ \left(\begin{smallmatrix} B & -C\\ C & B\end{smallmatrix}\right)\in\mathfrak{u}\}\\ & \ltimes(\mathcal{N}^1_{1,\dots,k}+\mathcal{N}^2_{1,\dots,k}+\mathcal{N}^1_{l+1,\dots,n}+\mathcal{C})\end{aligned}$$

$$= \left\{\left.\begin{pmatrix} 0&0&-z_1^t&-\psi_1(A)^t&-z_1'^t&-z_2^t&-\psi_2(A)^t&-\psi_3(A)^t&0&-c\\ 0&0&z_2^t&\psi_2(A)^t&\psi_3(A)^t&-z_1^t&-\psi_1(A)^t&-z_1'^t&c&0\\ 0&0&B&0&0&-C&0&0&z_1&-z_2\\ 0&0&0&0&0&0&0&0&\psi_1(A)&-\psi_2(A)\\ 0&0&0&0&0&0&0&0&z_1'&-\psi_3(A)\\ 0&0&C&0&0&B&0&&z_2&z_1\\ 0&0&0&0&0&0&0&0&\psi_2(A)&\psi_1(A)\\ 0&0&0&0&0&0&0&0&\psi_3(A)&z_1'\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{pmatrix}\right| \begin{array}{c} c\in\mathbb{R},\\ z_1,z_2\in\mathbb{R}^k,\\ z_1'\in\mathbb{R}^{n-l},\\ A=\\ \left(\begin{smallmatrix} B & -C\\ C & B\end{smallmatrix}\right)\\ \in\mathfrak{u}\end{array}\right\},$$

where $0<k\le l\le n$, $\mathfrak{u}\subset\mathfrak{u}(k)$ is a subalgebra such that $\dim\mathfrak{z}(\mathfrak{u})\ge n+l-2k$, $\psi:\mathfrak{u}\to E^1_{k+1,\dots,l}\oplus E^2_{k+1,\dots,l}\oplus E^2_{l+1,\dots,n}$ is a surjective linear map with $\psi|_{\mathfrak{u}'}=0$, $\psi_1=\mathrm{pr}_{E^1_{k+1,\dots,l}}\circ\psi$, $\psi_2=\mathrm{pr}_{E^2_{k+1,\dots,l}}\circ\psi$ and $\psi_3=\mathrm{pr}_{E^2_{l+1,\dots,n}}\circ\psi$;

$$\begin{aligned}\mathfrak{hol}^{m,\mathfrak{u},\psi,k,l,r} = {} & \{(0,0,B,C,\psi_1(B,C)+\psi_4(B,C),\psi_2(B,C)+\psi_3(B,C),0)|\ \left(\begin{smallmatrix} B & -C\\ C & B\end{smallmatrix}\right)\in\mathfrak{u}\}\\ & \ltimes(\mathcal{N}^1_{1,\dots,k}+\mathcal{N}^2_{1,\dots,k}+\mathcal{N}^1_{l+1,\dots,m}+\mathcal{N}^1_{m+1,\dots,r}+\mathcal{C})\end{aligned}$$

$$= \left\{\left.\begin{pmatrix} 0&0&-z_1^t&-\psi_1(A)^t&-z_1'^t&-z_1''^t&-\psi_4(A)^t&-z_2^t&-\psi_2(A)^t&-\psi_3(A)^t&0&0&0&-c\\ 0&0&z_2^t&\psi_2(A)^t&\psi_3(A)^t&0&0&-z_1^t&-\psi_1(A)^t&-z_1'^t&-z_1''^t&-\psi_4(A)^t&c&0\\ 0&0&B&0&0&0&0&-C&0&0&0&0&z_1&-z_2\\ 0&0&0&0&0&0&0&0&0&0&0&\psi_1(A)&-\psi_2(A)&\\ 0&0&0&0&0&0&0&0&0&0&0&0&z_1'&-\psi_3(A)\\ 0&0&0&0&0&0&0&0&0&0&0&0&z_1''&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&\psi_4(A)&0\\ 0&0&C&0&0&0&0&B&0&0&0&0&z_2&z_1\\ 0&0&0&0&0&0&0&0&0&0&0&0&\psi_2(A)&\psi_1(A)\\ 0&0&0&0&0&0&0&0&0&0&0&0&\psi_3(A)&z_1'\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&z_1''\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&\psi_4(A)\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}\right| \begin{array}{c} c\in\mathbb{R},\\ z_1,z_2\in\mathbb{R}^k,\\ z_1'\in\mathbb{R}^{m-l},\\ z_1''\in\mathbb{R}^{r-m},\\ \\ A=\\ \left(\begin{smallmatrix} B & -C\\ C & B\end{smallmatrix}\right)\\ \in\mathfrak{u}\end{array}\right\},$$

where $0<k\le l\le m\le r\le n$, $0<m<n$, $\mathfrak{u}\subset\mathfrak{u}(k)$ is a subalgebra such that $\dim\mathfrak{z}(\mathfrak{u})\ge(n+m+l-2k-r)$, $\psi:\mathfrak{u}\to E^1_{k+1,\dots,l}\oplus E^2_{k+1,\dots,l}\oplus E^2_{l+1,\dots,m}\oplus E^1_{r+1,\dots,n}$ is a surjective linear map with $\psi|_{\mathfrak{u}'}=0$, $\psi_1=\mathrm{pr}_{E^1_{k+1,\dots,l}}\circ\psi$, $\psi_2=\mathrm{pr}_{E^2_{k+1,\dots,l}}\circ\psi$, $\psi_3=\mathrm{pr}_{E^2_{l+1,\dots,m}}\circ\psi$, and $\psi_4=\mathrm{pr}_{E^1_{r+1,\dots,n}}\circ\psi$.

We see that to each weakly-irreducible not irreducible holonomy algebras $\mathfrak{hol} \subset \mathfrak{u}(1, n+1)_{<\mathbb{R}p_1,\mathbb{R}p_2>}$ an integer $0 \leq m \leq n$ and a subalgebra $\mathfrak{u} = \mathrm{pr}_{\mathfrak{u}(m)} \mathfrak{hol} \subset \mathfrak{u}(m)$ are associated. Recall that a pseudo-Kählerian manifold is called *special pseudo-Kählerian* if its Ricci tensor is zero. This is equivalent to the inclusion $\mathfrak{hol}_x \subset \mathfrak{su}(T_x M, g_x, J_x^M)$.

Corollary 3.1. *1) A subalgebra $\mathfrak{g} \subset \mathfrak{su}(1,1)$ is the weakly-irreducible not irreducible holonomy algebra of a special pseudo-Kählerian manifold of signature $(2,2)$ if and only if $\mathfrak{g}$ is conjugated to the subalgebra $\mathcal{C} \subset \mathfrak{su}(1,1)_{<p_1,p_2>}$ or to $\mathfrak{su}(1,1)_{<p_1,p_2>}$.*

2) Let $n \geq 1$. Then a subalgebra $\mathfrak{g} \subset \mathfrak{su}(1, n+1)$ is the weakly-irreducible not irreducible holonomy algebra of a special pseudo-Kählerian manifold of signature $(2, 2n+2)$ if and only if $\mathfrak{g}$ is conjugated to one of the following subalgebras of $\mathfrak{su}(1, n+1)_{<p_1,p_2>}$:

$\mathfrak{hol}^{m,\mathfrak{u},\mathcal{A}^1,\phi}$, $\mathfrak{hol}^{m,\mathfrak{u},\varphi,\phi}$ *with* $\phi(B,C) = -\frac{1}{n-m+2}\operatorname{tr} C$*;*

$\mathfrak{hol}^{n,\mathfrak{u},\psi,k,l}$, $\mathfrak{hol}^{m,\mathfrak{u},\psi,k,l,r}$ *with* $\mathfrak{u} \subset \mathfrak{su}(k)$*.*

Now we construct an example of metric with the holonomy algebra $\mathfrak{hol}$ for each Lie algebra from Theorem 3.1.

Let $0 \leq m \leq n$ and $\mathfrak{u} \subset \mathfrak{u}(m)$ be a subalgebra. Denote by $L \subset E$ the vector subspace annihilated by $\mathfrak{u}$. We can choose the basis $e_1, ..., e_n, f_1, ..., f_n$ in such a way that $L = \mathrm{span}\{e_{n_0+1}, ..., e_n, f_{n_0+1}, ..., f_n\}$, where $\dim L = 2(n - n_0)$. Then $\mathfrak{u} \subset \mathfrak{u}(n_0)$ and $\mathfrak{u}$ does not annihilate any proper subspace of $E_{1,...,n_0}$. Let us consider a basis $A_1 = \left(\begin{smallmatrix} B_1 & -C_1 \\ C_1 & B_1 \end{smallmatrix}\right)$,...,$A_N = \left(\begin{smallmatrix} B_N & -C_N \\ C_N & B_N \end{smallmatrix}\right)$ of the vector space $\mathfrak{u}$ such that $A_1, ..., A_{N_1}$ is a basis of the vector space $\mathfrak{u}'$ and $A_{N_1+1}, ..., A_N$ is a basis of the vector space $\mathfrak{z}(\mathfrak{u})$ ($N = \dim \mathfrak{u}$, $N_1 = \dim \mathfrak{u}'$). We denote by $(B^i_{\alpha j})_{i,j=1}^{n_0}$ and $(C^i_{\alpha j})_{i,j=1}^{n_0}$ the elements of the matrices B_α and C_α, respectively, where $\alpha = 1, ..., N$.

Let $x^1, ..., x^{2n+4}$ be the standard coordinates on $\mathbb{R}^{2n+4}$. Consider the following metric on $\mathbb{R}^{2n+4}$:

$$g = 2dx^1 dx^{2n+3} + 2dx^2 dx^{2n+4} + \sum_{i=3}^{2n+2} (dx^i)^2 + 2\sum_{i=3}^{2n+2} u^i dx^i dx^{2n+4} \\ + f_1 \cdot (dx^{2n+3})^2 + f_2 \cdot (dx^{2n+4})^2 + 2f_3 dx^{2n+3} dx^{2n+4}, \quad (1)$$

where u^3,...,u^{2n+2}, f_1, f_2 and f_3 are some functions which depend on the holonomy algebra that we wish to obtain.

For the linear maps $\varphi, \phi : \mathfrak{u} \to \mathbb{R}$ we define the numbers $\varphi_\alpha = \varphi(A_\alpha)$ and $\phi_\alpha = \phi(A_\alpha)$, $\alpha = N_1 + 1, ..., N$. For the linear map $\psi_1 : \mathfrak{u} \to E^1_{k+1,...,l}$ we define the numbers $\psi_{1\alpha i}$ such

that $\psi_1(A_\alpha) = \sum_{i=k+1}^{l} \psi_{1\alpha i} e_i$, $\alpha = N_1+1, ..., N$. We define analogous numbers for the linear maps ψ_2, ψ_3 and ψ_4.

Define the following functions

$$
\begin{aligned}
f_1^0 =& \sum_{\alpha=1}^{N} \frac{(x^{2n+3})^{\alpha-1}}{(\alpha-1)!} \left(\sum_{i,j=1}^{n_0} \left(B_{\alpha j}^i x^{i+2} x^{n+j+2} + \frac{1}{2} C_{\alpha j}^i x^{i+2} x^{j+2} + \frac{1}{2} C_{\alpha j}^i x^{n+i+2} x^{n+j+2} \right) \right), \\
f_2^0 =& f_1^0 + \sum_{i=1}^{n_0} \left(\left(\sum_{\alpha=1}^{N} \frac{1}{\alpha!} \sum_{j=1}^{n_0} \left(B_{\alpha j}^i x^{j+2} - C_{\alpha j}^i x^{n+j+2} \right) (x^{2n+3})^\alpha \right)^2 + \right. \\
& \left. \left(\sum_{\alpha=1}^{N} \frac{1}{\alpha!} \sum_{j=1}^{n_0} \left(B_{\alpha j}^i x^{n+j+2} + C_{\alpha j}^i x^{j+2} \right) (x^{2n+3})^\alpha \right)^2 \right), \\
f_3^0 =& 0.
\end{aligned}
$$

For the maps φ and ϕ define the functions

$$
\begin{aligned}
f_1^\varphi =& -2 \sum_{\alpha=N_1+1}^{N} \frac{1}{\alpha!} \varphi_\alpha x^2 (x^{2n+3})^\alpha, \quad f_2^\varphi = -f_1^\varphi, \quad f_3^\varphi = 2 \sum_{\alpha=N_1+1}^{N} \frac{1}{\alpha!} \varphi_\alpha x^1 (x^{2n+3})^\alpha, \\
f_1^\phi =& \sum_{\alpha=N_1+1}^{N} \phi_\alpha \frac{(x^{2n+3})^{\alpha-1}}{(\alpha-1)!} \left(-\frac{2}{\alpha} x^1 x^{2n+3} + \sum_{i=m+1}^{n} (x^{i+2})^2 \right),
\end{aligned}
$$

$$
\begin{aligned}
f_2^\phi =& \sum_{\alpha=N_1+1}^{N} \phi_\alpha \frac{(x^{2n+3})^{\alpha-1}}{(\alpha-1)!} \left(\frac{2}{\alpha} x^1 x^{2n+3} + \sum_{i=m+1}^{n} (x^{n+i+2})^2 \right. \\
& \left. + \frac{(x^{2n+3})^2}{(\alpha+1)\alpha} \sum_{i=m+1}^{n} ((x^{i+2})^2 + (x^{n+i+2})^2) \right), \\
f_3^\phi =& \sum_{\alpha=N_1+1}^{N} \phi_\alpha \frac{(x^{2n+3})^{\alpha-1}}{(\alpha-1)!} \left(-\frac{2}{\alpha} x^2 x^{2n+3} + \sum_{i=m+1}^{n} x^{i+2} x^{n+i+2} \right)
\end{aligned}
$$

For $K=N+1,N+2$ we define the functions

$$
\begin{aligned}
f_1^{\mathcal{A}^1,K} &= -2\frac{1}{K!}x^2(x^{2n+3})^K, \quad f_2^{\mathcal{A}^1,K} = -f_1^{\mathcal{A}^1,K}, \quad f_3^{\mathcal{A}^1,K} = 2\frac{1}{K!}x^1(x^{2n+3})^K,\\
f_1^{\tilde{\mathcal{A}}^2,K} &= \frac{1}{(K-1)!}(x^{2n+3})^{K-1}\left(-\frac{2}{K}x^1x^{2n+3}+\sum_{i=m+1}^{n}(x^{i+2})^2\right),\\
f_2^{\tilde{\mathcal{A}}^2,K} &= \frac{1}{(K-1)!}(x^{2n+3})^{K-1}\left(\frac{2}{K}x^1x^{2n+3}+\sum_{i=m+1}^{n}(x^{n+i+2})^2\right.\\
&\qquad\qquad\qquad\qquad\left.+\frac{(x^{2n+3})^2}{(\alpha+1)\alpha}\sum_{i=m+1}^{n}((x^{i+2})^2+(x^{n+i+2})^2)\right),\\
f_3^{\tilde{\mathcal{A}}^2,K} &= \frac{1}{(K-1)!}(x^{2n+3})^{K-1}\left(-\frac{2}{K}x^2x^{2n+3}+\sum_{i=m+1}^{n}x^{i+2}x^{n+i+2}\right).
\end{aligned}
$$

For any numbers $1\leq m_1\leq m_2\leq n$ consider the functions

$$
\tilde{f}_{1m_1}^{m_2} = \sum_{i=m_1}^{m_2}(x_{i+2}^2-x_{n+i+2}^2), \quad \tilde{f}_{2m_1}^{m_2} = -\tilde{f}_{1m_1}^{m_2}, \quad \tilde{f}_{3m_1}^{m_2} = 2\sum_{i=m_1}^{m_2}x^{i+2}x^{n+i+2}.
$$

For any numbers $0\leq m_1\leq m_2\leq n$ and $K\geq N+1$ consider the functions

$$
\begin{aligned}
\breve{f}_{1m_1}^{Km_2} &= -\sum_{i=m_1}^{m_2}\frac{2}{(K+i-m_1)!}x^{n+i+2}(x^{2n+3})^{K+i-m_1}, \quad \breve{f}_{2m_1}^{Km_2} = -\breve{f}_{1m_1}^{Km_2},\\
\breve{f}_{3m_1}^{Km_2} &= -\sum_{i=m_1}^{m_2}\frac{2}{(K+i-m_1)!}x^{i+2}(x^{2n+3})^{K+i-m_1}.
\end{aligned}
$$

For the Lie algebra $\mathfrak{hol}^{n,\mathfrak{u},\psi,k,l}$ we consider the functions

$$
\begin{aligned}
f_1^{n,\psi} &= \sum_{\alpha=N_1+1}^{N}\frac{2}{\alpha!}\left(\sum_{i=k+3}^{l+2}\psi_{1\alpha i}x^{n+i}-\sum_{i=k+3}^{l+2}\psi_{2\alpha i}x^{i}-\sum_{i=l+3}^{n+2}\psi_{3\alpha i}x^{i}\right)(x^{2n+3})^{\alpha},\\
f_2^{n,\psi} &= -f_1^{n,\psi},\\
f_3^{n,\psi} &= \sum_{\alpha=N_1+1}^{N}\frac{2}{\alpha!}\left(-\sum_{i=k+3}^{l+2}\psi_{1\alpha i}x^{i}-\sum_{i=k+3}^{l+2}\psi_{2\alpha i}x^{n+i}-\sum_{i=l+3}^{n+2}\psi_{3\alpha i}x^{n+i}\right)(x^{2n+3})^{\alpha}.
\end{aligned}
$$

For the Lie algebra $\mathfrak{hol}^{m,\mathfrak{u},\psi,k,l,r}$ we define the functions

$$\begin{aligned} f_1^{m,\psi} =& \sum_{\alpha=N_1+1}^{N} \frac{2}{\alpha!} \left(\sum_{i=k+3}^{l+2} \psi_{1\alpha i} x^{n+i} + \sum_{i=r+3}^{n+2} \psi_{4\alpha i} x^{n+i} \right. \\ & \left. - \sum_{i=k+3}^{l+2} \psi_{2\alpha i} x^{i} - \sum_{i=l+3}^{m+2} \psi_{3\alpha i} x^{i} \right) (x^{2n+3})^{\alpha}, \\ f_2^{m,\psi} =& - f_1^{m,\psi}, \\ f_3^{m,\psi} =& \sum_{\alpha=N_1+1}^{N} \frac{2}{\alpha!} \left(- \sum_{i=k+3}^{l+2} \psi_{1\alpha i} x^{i} - \sum_{i=r+3}^{n+2} \psi_{4\alpha i} x^{i} \right. \\ & \left. - \sum_{i=k+3}^{l+2} \psi_{2\alpha i} x^{n+i} - \sum_{i=l+3}^{m+2} \psi_{3\alpha i} x^{n+i} \right) (x^{2n+3})^{\alpha}. \end{aligned}$$

Define the functions $u^3,\dots,u^{n_0+2}$, $u^{n+3},\dots,u^{n+n_0+2}$ as follows:

$$\begin{aligned} u^i =& \sum_{\alpha=1}^{N} \frac{1}{\alpha!} \left(\sum_{j=1}^{n_0} (B^{i-2}_{\alpha j} x^{j+2} - C^{i-2}_{\alpha j} x^{n+j+2}) \right) (x^{2n+3})^{\alpha}, \\ u^{n+i} =& \sum_{\alpha=1}^{N} \frac{1}{\alpha!} \left(\sum_{j=1}^{n_0} (B^{i-2}_{\alpha j} x^{n+j+2} + C^{i-2}_{\alpha j} x^{j+2}) \right) (x^{2n+3})^{\alpha}, \end{aligned}$$

where $3 \leq i \leq n_0 + 2$.

For the Lie algebras $\mathfrak{hol}^{m,\mathfrak{u},\mathcal{A}^1,\tilde{\mathcal{A}}^2}$ and $\mathfrak{hol}^{m,\mathfrak{u},\varphi,\tilde{\mathcal{A}}^2}$ we set in addition

$$\begin{aligned} u^i =& - \frac{1}{(N+2)!} x^{n+i} (x^{2n+3})^{N+2}, \\ u^{n+i} =& \frac{1}{(N+2)!} x^{i} (x^{2n+3})^{N+2}, \end{aligned}$$

where $m + 3 \leq i \leq 2n + 2$.

For the Lie algebras $\mathfrak{hol}^{m,\mathfrak{u},\mathcal{A}^1,\phi}$ and $\mathfrak{hol}^{m,\mathfrak{u},\varphi,\phi}$ we set

$$\begin{aligned} u^i =& - \sum_{\alpha=1}^{N} \frac{1}{\alpha!} \phi_\alpha x^{n+i} (x^{2n+3})^{\alpha}, \\ u^{n+i} =& \sum_{\alpha=1}^{N} \frac{1}{\alpha!} \phi_\alpha x^{i} (x^{2n+3})^{\alpha}, \end{aligned}$$

where $m + 3 \leq i \leq 2n + 2$.

For the Lie algebra $\mathfrak{hol}^{m,\mathfrak{u},\lambda}$ we set

$$\begin{aligned} u^i =& - \frac{1}{(N+1)!} \lambda x^{n+i} (x^{2n+3})^{N+1}, \\ u^{n+i} =& \frac{1}{(N+1)!} \lambda x^{i} (x^{2n+3})^{N+1}, \end{aligned}$$

where $m+3 \leq i \leq 2n+2$.

We assume that the functions u^i that were not defined now are zero.

If we choose the functions f_1, f_2 and f_3 such that $f_1(0) = f_2(0) = f_3(0) = 0$, then for the metric g given by (1) we have $g_0 = \eta$ and we can identify the tangent space to $\mathbb{R}^{2n+4}$ at 0 with the vector space $\mathbb{R}^{2,2n+2}$.

Note that if $n = 0$, then

$$g = 2dx^1dx^3 + 2dx^2dx^4 + f_1 \cdot (dx^3)^2 + f_2 \cdot (dx^4)^2 + 2f_3dx^3dx^4.$$

Theorem 3.2. *Let $\mathfrak{hol}_0$ be the holonomy algebra of the metric g at the point $0 \in \mathbb{R}^{2n+4}$.*

1) *Let $n = 0$, then $\mathfrak{hol}_0$ depends on the functions f_1, f_2 and f_3 as in Table 3.1.*

Table 3.1. *Dependence of $\mathfrak{hol}_0$ on the functions f_1, f_2 and f_3 for $n = 0$*

f_i, $(i = 1,2,3)$	$\mathfrak{hol}_0$
$f_1 = -2x^2x^3 - x^1(x^3)^2$, $f_2 = -f_1$, $f_3 = 2x^1x^3 - x^2(x^3)^2$	$\mathfrak{hol}^1_{n=0}$
$f_1 = (x^1)^2 - (x^2)^2$, $f_2 = -f_1$, $f_3 = 2x^1x^2$	$\mathfrak{hol}^2_{n=0}$
$f_1 = -2\gamma_1x^2x^3 - 2\gamma_2x^1x^3$, $f_2 = -f_1$, $f_3 = 2\gamma_1x^1x^3 - 2\gamma_2x^2x^3$	$\mathfrak{hol}^{\gamma_1,\gamma_2}_{n=0}$ *(if $\gamma_1^2 + \gamma_2^2 \neq 0$)*
$f_1 = (x^4)^2$, $f_2 = f_3 = 0$	$\mathfrak{hol}^{\gamma_1=0,\gamma_2=0}_{n=0}$

2) *Let $n > 0$, then $\mathfrak{hol}_0$ depends on the functions f_1, f_2 and f_3 as in Table 3.2.*

Table 3.2. *Dependence of $\mathfrak{hol}_0$ on the functions f_1, f_2 and f_3 for $n > 0$*

f_i, $(i = 1,2,3)$	$\mathfrak{hol}_0$
$f_i = f_i^{\mathcal{A}^1,N+1} + f_i^{\tilde{\mathcal{A}}^2,N+2} + f_i^0 + \tilde{f}^m_{i\,n_0+1} + \breve{f}^{N+3\ n}_{i\,m+1}$	$\mathfrak{hol}^{m,\mathfrak{u},\mathcal{A}^1,\tilde{\mathcal{A}}^2}$
$f_i = f_i^{\mathcal{A}^1,N+1} + f_i^{\phi} + f_i^0 + \tilde{f}^m_{i\,n_0+1} + \breve{f}^{N+2\ n}_{i\,m+1}$	$\mathfrak{hol}^{m,\mathfrak{u},\mathcal{A}^1,\phi}$
$f_i = f_i^{\varphi} + f_i^{\tilde{\mathcal{A}}^2,N+1} + f_i^0 + \tilde{f}^m_{i\,n_0+1} + \breve{f}^{N+2\ n}_{i\,m+1}$	$\mathfrak{hol}^{m,\mathfrak{u},\varphi,\tilde{\mathcal{A}}^2}$
$f_i = f_i^{\varphi} + f_i^{\phi} + f_i^0 + \tilde{f}^m_{i\,n_0+1} + \breve{f}^{N+2\ n}_{i\,m+1}\,\breve{f}^{N+1\ n}_{i\,m+1}$	$\mathfrak{hol}^{m,\mathfrak{u},\varphi,\phi}$
$f_i = f_i^{\mathcal{A}^1,N+1} + \lambda f_i^{\tilde{\mathcal{A}}^2,N+1} + f_i^0 + \tilde{f}^m_{i\,n_0+1} + \breve{f}^{N+2\ n}_{i\,m+1}$	$\mathfrak{hol}^{m,\mathfrak{u},\lambda}$
$f_i = f_i^0 + \tilde{f}^k_{i\,n_0+1} + f_i^{n,\psi} + \breve{f}^{N+1\,n}_{i\,l+1}$	$\mathfrak{hol}^{n,\mathfrak{u},\psi,k,l}$ *(if $\dim \mathfrak{z}(\mathfrak{u}) \geq n + l - 2k$)*
$f_i = f_i^0 + \tilde{f}^k_{i\,n_0+1} + f_i^{m,\psi} + \breve{f}^{N+1\,r}_{i\,l+1}$	$\mathfrak{hol}^{m,\mathfrak{u},\psi,k,l,r}$ *(if $\dim \mathfrak{z}(\mathfrak{u}) \geq n + m + l - 2k - r$)*

3.2 Proof of Theorem 3.1

In this section we will prove that the Lie algebras of Theorem 3.1 exhaust all weakly-irreducible Berger subalgebras of $\mathfrak{u}(1,n+1)_{<p_1,p_2>}$, i.e. all candidates for the holonomy algebras. The rest of the proof of Theorem 3.1 will follow from Theorem 3.2.

Now we will describe the spaces of curvature tensors $\mathcal{R}(\mathfrak{g})$ for subalgebras $\mathfrak{g}\subset\mathfrak{u}(1,n+1)_{<p_1,p_2>}$. We will use the following obvious fact. Let $\mathfrak{f}_1\subset\mathfrak{f}_2\subset\mathfrak{so}(r,s)$, then

$$R\in\mathcal{R}(\mathfrak{f}_1) \text{ if and only if } R\in\mathcal{R}(\mathfrak{f}_2) \text{ and } R(\mathbb{R}^{r,s}\wedge\mathbb{R}^{r,s})\subset\mathfrak{f}_1. \qquad (2)$$

First we will describe the space $\mathcal{R}(\mathfrak{g}^{\mathfrak{u}})$ for the Lie algebra

$$\mathfrak{g}^{\mathfrak{u}}=\left\{\left.\begin{pmatrix} a_1 & -a_2 & -X^t & 0 & -c\\ a_2 & a_1 & -Y^t & c & 0\\ 0 & 0 & A & X & Y\\ 0 & 0 & 0 & -a_1 & -a_2\\ 0 & 0 & 0 & a_2 & -a_1\end{pmatrix}\right|\begin{matrix} a_1,a_2,c\in\mathbb{R},\\ X,Y\in\mathbb{R}^{2n},\\ A\in\mathfrak{u}\end{matrix}\right\}\subset\mathfrak{so}(2,2n+2)_{<p_1,p_2>}.$$

Here $\mathfrak{u}\subset\mathfrak{u}(n)$ is a subalgebra and $\mathfrak{so}(2,2n+2)_{<p_1,p_2>}$ is the subalgebra of $\mathfrak{so}(2,2n+2)$ that preserves the isotropic plane $\mathbb{R}p_1\oplus\mathbb{R}p_2$.

Using the form η, we identify $\mathfrak{so}(2,2n+2)$ with the space

$$\mathbb{R}^{2,2n+2}\wedge\mathbb{R}^{2,2n+2}=\operatorname{span}\{u\wedge v=u\otimes v-v\otimes u|u,v\in\mathbb{R}^{2,2n+2}\}.$$

The identification is given by the formula

$$(u\wedge v)w=\eta(u,w)v-\eta(v,w)u \text{ for all } u,v,w\in\mathbb{R}^{2,2n+2}.$$

Similarly, we identify $\mathfrak{so}(n)$ with $E\wedge E\subset\mathbb{R}^{2,2n+2}\wedge\mathbb{R}^{2,2n+2}$. It is easy to see that the element

$$\begin{pmatrix} a_1 & -a_2 & -X^t & 0 & -c\\ a_2 & a_1 & -Y^t & c & 0\\ 0 & 0 & A & X & Y\\ 0 & 0 & 0 & -a_1 & -a_2\\ 0 & 0 & 0 & a_2 & -a_1\end{pmatrix}\in\mathfrak{g}^{\mathfrak{u}}$$

corresponds to

$$-a_1(p_1\wedge q_1+p_2\wedge q_2)+a_2(p_1\wedge q_2-p_2\wedge q_1)+A+p_1\wedge X+p_2\wedge Y+cp_1\wedge p_2\in\mathbb{R}^{2,2n+2}\wedge\mathbb{R}^{2,2n+2}.$$

Thus we obtain the following decomposition of $\mathfrak{g}^{\mathfrak{u}}$:

$$\mathfrak{g}^{\mathfrak{u}}=(\mathbb{R}(p_1\wedge q_1+p_2\wedge q_2)\oplus\mathbb{R}(p_1\wedge q_2-p_2\wedge q_1)\oplus\mathfrak{u})\ltimes(p_1\wedge E+p_2\wedge E+\mathbb{R}p_1\wedge p_2). \quad (3)$$

By analogy, we have

$$\begin{aligned}\mathfrak{u}(1,n+1)_{<p_1,p_2>} = \quad & (\mathbb{R}(p_1\wedge q_1+p_2\wedge q_2)\oplus\mathbb{R}(p_1\wedge q_2-p_2\wedge q_1)\oplus\mathfrak{u}(n))\\ & \ltimes(\{p_1\wedge x+p_2\wedge Jx|x\in E^1\}+\{p_1\wedge Jx-p_2\wedge x|x\in E^1\}+\mathbb{R}p_1\wedge p_2).\end{aligned}$$

The decomposition $\mathbb{R}^{2,2n+2}=\mathbb{R}p_1+\mathbb{R}p_2+E+\mathbb{R}q_1+\mathbb{R}q_2$ gives us the decomposition

$$\begin{aligned}\mathbb{R}^{2,2n+2}\wedge\mathbb{R}^{2,2n+2} &= \mathbb{R}(p_1\wedge q_1+p_2\wedge q_2)+\mathbb{R}(p_1\wedge q_2-p_2\wedge q_1)+\mathbb{R}(p_1\wedge q_1-p_2\wedge q_2)+\mathbb{R}(p_1\wedge q_2+p_2\wedge q_1)\\ &+E\wedge E+p_1\wedge E+p_2\wedge E+q_1\wedge E+q_2\wedge E+\mathbb{R}p_1\wedge p_2+\mathbb{R}q_1\wedge q_2. \quad (4)\end{aligned}$$

The metric η defines the metric $\eta\wedge\eta$ on $\mathbb{R}^{2,2n+2}\wedge\mathbb{R}^{2,2n+2}$. Let $R\in\mathcal{R}(\mathfrak{g}^{\mathfrak{u}})$. It can be proved that

$$\eta\wedge\eta(R(u\wedge v),z\wedge w)=\eta\wedge\eta(R(z\wedge w),u\wedge v) \text{ for all } u,v,z,w\in\mathbb{R}^{2,2n+2}. \qquad (5)$$

This shows that $R:\mathbb{R}^{2,2n+2}\wedge\mathbb{R}^{2,2n+2}\to\mathfrak{g}^{\mathfrak{u}}\subset\mathbb{R}^{2,2n+2}\wedge\mathbb{R}^{2,2n+2}$ is a symmetric linear map. Hence R is zero on the orthogonal complement to $\mathfrak{g}^{\mathfrak{u}}$ in $\mathbb{R}^{2,2n+2}\wedge\mathbb{R}^{2,2n+2}$. In particular,

$$R|_{\mathbb{R}(p_1\wedge q_1-p_2\wedge q_2)+\mathbb{R}(p_1\wedge q_2+p_2\wedge q_1)+\mathbb{R}p_1\wedge p_2+p_1\wedge E+p_2\wedge E}=0. \qquad (6)$$

Thus R can be considered as the linear map

$$\begin{aligned}R:\ & \mathbb{R}(p_1\wedge q_1+p_2\wedge q_2)+\mathbb{R}(p_1\wedge q_2-p_2\wedge q_1)+E\wedge E+q_1\wedge E+q_2\wedge E+\mathbb{R}q_1\wedge q_2\\ & \to\mathfrak{g}^{\mathfrak{u}}=(\mathbb{R}(p_1\wedge q_1+p_2\wedge q_2)\oplus\mathbb{R}(p_1\wedge q_2-p_2\wedge q_1)\oplus\mathfrak{u})\ltimes(p_1\wedge E+p_2\wedge E+\mathbb{R}p_1\wedge p_2).\end{aligned}$$

Consider the following set of subsets of $\mathbb{R}^{2,2n+2}\wedge\mathbb{R}^{2,2n+2}$,

$$\mathcal{F}=\{\mathbb{R}(p_1\wedge q_1+p_2\wedge q_2),\mathbb{R}(p_1\wedge q_2-p_2\wedge q_1),\mathfrak{u},p_1\wedge E,p_2\wedge E,\mathbb{R}p_1\wedge p_2\}.$$

For any $F\in\mathcal{F}$ we set

$$R_F=\mathrm{pr}_F\circ R:\mathbb{R}^{2,2n+2}\wedge\mathbb{R}^{2,2n+2}\to F,$$

where pr_F is the projection with respect to the decomposition (3). Obviously,

$$R=\sum_{F\in\mathcal{F}}R_F.$$

Note that from (6) it follows that $R(p_1\wedge q_1)=R(p_2\wedge q_2)$ and $R(p_1\wedge q_2)=-R(p_2\wedge q_1)$. Using the Bianchi identity, (5) and (6), it is easy to show that R can be found from Table 3.2.1 on page 47, where on the position $(u\wedge v,\mathcal{F})$ stays the value $R_F(u\wedge v)$.

In Table 3.2.1 we have $x, y \in E$, $\lambda_1, ..., \lambda_5 \in \mathbb{R}$, $K_1, K_2, L_1, L_2 \in \mathrm{Hom}(\mathbb{R}, E), R_{\mathfrak{u}} \in \mathcal{R}(\mathfrak{u})$,

$$P_1, P_2 \in \mathcal{P}(\mathfrak{u}) = \left\{ P \in \mathrm{Hom}(E, \mathfrak{u}) \left| \begin{array}{c} \eta(P(u)v, w) + \eta(P(v)w, u) + \eta(P(w)u, v) = 0 \\ \text{for all } u, v, w \in E \end{array} \right. \right\},$$

$T_1, T_2 \in \mathrm{Hom}(E, E)$, $T_1^* = T_1$, $T_2^* = T_2$ and $S \in \mathrm{Hom}(E, E)$ is a linear map such that $S - S^* \in \mathfrak{u}$.

It is easy to show that for any elements as above the linear map $R \in \mathrm{Hom}(\mathbb{R}^{2,2n+2} \wedge \mathbb{R}^{2,2n+2}, \mathfrak{g}^{\mathfrak{u}})$ defined by Table 3.2.1 and (6) satisfies $R \in \mathcal{R}(\mathfrak{g}^{\mathfrak{u}})$.

For any $0 \leq m \leq n$ and $\mathfrak{u} \subset \mathfrak{u}(m) \oplus \mathfrak{sod}(m+1, ..., n)$ consider the subalgebra

$$\begin{aligned} \mathfrak{g}^{m,\mathfrak{u}} = \quad & (\mathbb{R}(p_1 \wedge q_1 + p_2 \wedge q_2) \oplus \mathbb{R}(p_1 \wedge q_2 - p_2 \wedge q_1 + J_{m+1,\dots,n}) \oplus \mathfrak{u}) \\ & \ltimes (\{p_1 \wedge x + p_2 \wedge Jx | x \in E^1\} + \{p_1 \wedge Jx - p_2 \wedge x | x \in E^1_{1,\dots,m}\} + \mathbb{R} p_1 \wedge p_2) \\ & \subset \mathfrak{u}(1, n+1)_{<p_1,p_2>}. \end{aligned}$$

For the $\mathfrak{u}(n)$-projection of the Lie algebra $\mathfrak{g}^{m,\mathfrak{u}}$ we have $\mathrm{pr}_{\mathfrak{u}(n)} \mathfrak{g}^{m,\mathfrak{u}} = \mathfrak{u} \oplus \mathbb{R} J_{m+1,\dots,n}$. Let us consider a curvature tensor $R \in \mathcal{R}(\mathfrak{g}^{m,\mathfrak{u}})$. Since $\mathfrak{g}^{m,\mathfrak{u}} \subset \mathfrak{g}^{\mathfrak{u} \oplus \mathbb{R} J_{m+1,\dots n}}$, to decompose R we can use (2) and Table 3.2.1. For $K_1 \in \mathrm{Hom}(\mathbb{R}, E)$ let $K_1^1 = \mathrm{pr}_{E^1} \circ K_1$ and $K_1^2 = \mathrm{pr}_{E^2} \circ K_1$. Then $K_1 = K_1^1 + K_1^2$. For $S \in \mathrm{Hom}(E, E)$ let

$$S^{11} = \mathrm{pr}_{E^1} \circ S|_{E^1}, \quad S^{12} = \mathrm{pr}_{E^1} \circ S|_{E^2}, \quad S^{21} = \mathrm{pr}_{E^2} \circ S|_{E^1}, \quad S^{22} = \mathrm{pr}_{E^2} \circ S|_{E^2}$$

and extend these linear maps to E mapping the natural complement to zero. We get the decomposition $S = S^{11} + S^{12} + S^{21} + S^{22}$. For $P_1 \in \mathcal{P}(\mathfrak{u})$ let $Q_1 = P_1^* \in \mathrm{Hom}(\mathfrak{u}, E)$, $Q_1^1 = \mathrm{pr}_{E^1} \circ Q_1$ and $Q_1^2 = \mathrm{pr}_{E^2} \circ Q_1$. Consequently, $Q_1 = Q_1^1 + Q_1^2$. Consider the analogous decompositions for the elements $K_2, L_1, L_2 \in \mathrm{Hom}(\mathbb{R}, E)$, $Q_2 = P_2^* \in \mathrm{Hom}(\mathfrak{u}, E)$, $T_1, T_2, S^* \in \mathrm{Hom}(E, E)$. Using the condition $R(\mathbb{R}^{2,2n+2} \wedge \mathbb{R}^{2,2n+2}) \subset \mathfrak{g}^{m,\mathfrak{u}}$, we obtain

$$K_2^2 = JK_1^1, \quad K_2^1 = JK_1^2, \quad K_1^1(1) \in E^1_{1,\dots,m}, \quad K_1^2(1) \in E^2_{1,\dots,m} \tag{7}$$

$$Q_2^2 = JQ_1^1, \quad Q_2^1 = JQ_1^2, \quad Q_1^2(1) \in E^2_{1,\dots,m} \tag{8}$$

$$T_1^{11} = -JS^{21}, \quad T_1^{21} = -JS^{11},$$
$$S^{21}(E^1_{1,\dots,m}) \subset E^2_{1,\dots,m}, \quad S^{21}(E^1_{m+1,\dots,n}) \subset E^2_{m+1,\dots,n}, \tag{9}$$

$$T_1^{12} = -JS^{22}, \quad T_1^{22} = -JS^{12}, \quad S^{22}(E) \subset E^2_{1,\dots,m}, \tag{10}$$

$$T_2^{11} = JS^{*21}, \quad T_2^{21} = JS^{*11}, \quad S^{*21}(E) \subset E^2_{1,\dots,m}, \tag{11}$$

$$T_2^{12} = JS^{*22}, \quad T_2^{22} = JS^{*12}, \quad S^{*22}(E) \subset E^2_{1,\dots,m}, \tag{12}$$

$$L_2^2 = JL_1^1, \quad L_2^1 = JL_1^2, \quad L_1^2(1) \in E^2_{1,\dots,m}. \tag{13}$$

From (9), (10), (11), (12), and the fact that $T_1^* = T_1$ and $T_2^* = T_2$ it follows that

$$(S^{22})^* = JS^{11}J, \quad (S^{12})^* = JS^{12}J, \quad (S^{21})^* = JS^{21}J,$$

$$S^{11}(E^1) \subset E^1_{1,\dots,m}, \quad S^{12}(E^2) \subset E^1_{1,\dots,m}. \quad S^{21}(E^1_{1,\dots,m}) \subset E^2_{1,\dots,m}.$$

Since $\mathrm{pr}_{\mathbb{R}(p_1\wedge q_2 - p_2\wedge q_1)} R(q_1 \wedge q_2) = \lambda_3(p_1 \wedge q_2 - p_2 \wedge q_1)$, we see that $S^{21}|_{E^1_{m+1,\dots,n}} = \lambda_3 J|_{E^1_{m+1,\dots,n}}$.

Using (8), we get $Q_2 = JQ_1$, i.e. $P_2^* = JP_1^*$. Hence, $P_2 = P_1J^*$. Thus, $P_2 = -P_1J$.

Since $\mathrm{pr}_{\mathbb{R}(p_1\wedge q_2 - p_2\wedge q_1)} R(E\wedge E) = \{0\}$, from Table 3.2.1 it follows that $\mathrm{pr}_{\mathfrak{u}(n)} R(E\wedge E) \subset \mathfrak{u}$. It is well known that from the inclusion $\mathfrak{u} \subset \mathfrak{u}(m) \oplus \mathfrak{sod}(m+1,...,n)$ it follows that $\mathcal{R}(\mathfrak{u}) = \mathcal{R}(\mathfrak{u}\cap\mathfrak{u}(m)) \oplus \mathcal{R}(\mathfrak{u}\cap\mathfrak{sod}(m+1,...,n))$. Moreover, $\mathcal{R}(\mathfrak{sod}(m+1,...,n)) = \{0\}$. Therefore, $\mathrm{pr}_{\mathfrak{u}(n)} R(E\wedge E) \subset \mathfrak{u}\cap\mathfrak{u}(m)$. Thus for $R_0 = \mathrm{pr}_{\mathfrak{u}(n)} \circ R|_{E\wedge E}$ we get $R_0 \in \mathcal{R}(\mathfrak{u}\cap\mathfrak{u}(m))$.

Similarly, since $\mathrm{pr}_{\mathbb{R}(p_1\wedge q_2 - p_2\wedge q_1)} R(q_1 \wedge E^1_{1,\dots,m}) = \{0\}$, we see that $\mathrm{pr}_{\mathfrak{u}(n)} R(q_1\wedge E) \subset \mathfrak{u}$. In [11] it was proved that

$$\mathcal{P}(\mathfrak{u}) = \mathcal{P}(\mathfrak{u}\cap\mathfrak{u}(m)) \oplus \mathcal{P}(\mathfrak{u}\cap\mathfrak{sod}(m+1,...,n)) \quad \text{and} \quad \mathcal{P}(\mathfrak{sod}(m+1,...,n)) = \{0\}.$$

Thus, $P_1 \in \mathcal{P}(\mathfrak{u}\cap\mathfrak{u}(m))$.

From (5) it follows that

$$R|_{\mathbb{R}(p_1\wedge q_1 - p_2\wedge q_2)+\mathbb{R}(p_1\wedge q_2+p_2\wedge q_1)+\mathbb{R}p_1\wedge p_2+\{q_1\wedge x - q_2\wedge Jx|x\in E^1\}+\{q_1\wedge Jx+q_2\wedge x|x\in E^1_{1,\dots,m}\}} = 0, \quad (14)$$

$$R|_{q_1\wedge E^2_{m+1,\dots,n}+q_2\wedge E^1_{m+1,\dots,n}+p_1\wedge E+p_2\wedge E} = 0. \quad (15)$$

In particular, $R(q_2\wedge Jx) = R(q_1\wedge x)$ for all $x\in E^1$, and $R(q_1\wedge Jx) = R(q_2\wedge x)$ for all $x \in E^1_{1,\dots,m}$.

We set the following denotation: $N_1 = K_1^1$, $N_2 = JK_1^2$, $M_1 = \mathrm{pr}_{E^1_{1,\dots,m}} \circ L_1^1$, $M_2 = -JL_1^2$, $M_3 = \mathrm{pr}_{E^1_{m+1,\dots,n}} \circ L_1^1$, $P = P_1$, $S^{21} = \mathrm{pr}_{E^2_{1,\dots,m}} \circ S|_{E^1_{1,\dots,m}}$. Now the curvature tensor $R \in \mathcal{R}(\mathfrak{g}^{m,\mathfrak{u}})$ can be found as above from the conditions (14), (15) and Table 3.2.2. In this case we assume that

$$\begin{aligned} \mathcal{F} = \; & \{\mathbb{R}(p_1\wedge q_1 + p_2\wedge q_2), \mathbb{R}(p_1\wedge q_2 - p_2\wedge q_1), \mathbb{R}J_{m+1,\dots,n}, \mathfrak{u}, p_1\wedge E, \\ & \{p_1\wedge x + p_2\wedge Jx|x\in E^1_{1,\dots,m}\}, \quad \{p_1\wedge Jx - p_2\wedge x|x\in E^1_{1,\dots,m}\}, \\ & \{p_1\wedge x + p_2\wedge Jx|x\in E^1_{m+1,\dots,n}\}, \mathbb{R}p_1\wedge p_2\}. \end{aligned}$$

In Table 3.2.2 we have $x_1, y_1 \in E$, $x\in E^1_{1,\dots,m}$, $y\in E^1_{m+1,\dots,n}$, $\lambda_1,...,\lambda_5\in\mathbb{R}$, $N_1, N_2, M_1, M_2 \in \mathrm{Hom}(\mathbb{R}, E^1_{1,\dots,m})$, $M_3 \in \mathrm{Hom}(\mathbb{R}, E^1_{m+1,\dots,n})$, if $m<n$, then $\lambda_1 = \lambda_2 = 0$ and $N_1 = N_2 =$

Table 3.2.1. Decomposition of a curvature tensor $R \in \mathcal{R}(\mathfrak{g}^{\mathfrak{u}})$

$u\wedge v \diagdown F$	$\mathbb{R}(p_1 \wedge q_1 + p_2 \wedge q_2)$	$\mathbb{R}(p_1 \wedge q_2 - p_2 \wedge q_1)$	$\mathfrak{u}$	$p_1 \wedge E$	$p_2 \wedge E$	$\mathbb{R}p_1 \wedge p_2$
$p_1 \wedge q_1$	$\lambda_1(p_1 \wedge q_1 + p_2 \wedge q_2)$	$\lambda_2(p_1 \wedge q_2 - p_2 \wedge q_1)$	0	$p_1 \wedge K_1(1)$	$p_2 \wedge K_2(1)$	$\lambda_4 p_1 \wedge p_2$
$p_1 \wedge q_2$	$-\lambda_2(p_1 \wedge q_1 + p_2 \wedge q_2)$	$\lambda_1(p_1 \wedge q_2 - p_2 \wedge q_1)$	0	$p_1 \wedge K_2(1)$	$-p_2 \wedge K_1(1)$	$\lambda_3 p_1 \wedge p_2$
$x \wedge y$	0	0	$R_{\mathfrak{u}}(x \wedge y)$	$\frac{1}{2} p_1 \wedge P_1^*(x \wedge y)$	$\frac{1}{2} p_2 \wedge P_2^*(x \wedge y)$	$\big(\eta(S(x), y) - \eta(S^*(x), y)\big) \cdot p_1 \wedge p_2$
$q_1 \wedge x$	$-K_1^*(x)(p_1 \wedge q_1 + p_2 \wedge q_2)$	$K_2^*(x)(p_1 \wedge q_2 - p_2 \wedge q_1)$	$P_1(x)$	$p_1 \wedge T_1(x)$	$p_2 \wedge S(x)$	$L_1^*(x) p_1 \wedge p_2$
$q_2 \wedge x$	$-K_2^*(x)(p_1 \wedge q_1 + p_2 \wedge q_2)$	$-K_1^*(x)(p_1 \wedge q_2 + p_2 \wedge q_1)$	$P_2(x)$	$p_1 \wedge S^*(x)$	$p_2 \wedge T_2(x)$	$L_2^*(x) p_1 \wedge p_2$
$q_1 \wedge q_2$	$-\lambda_4(p_1 \wedge q_1 + p_2 \wedge q_2)$	$\lambda_3(p_1 \wedge q_2 - p_2 \wedge q_1)$	$S - S^*$	$p_1 \wedge L_1(1)$	$p_2 \wedge L_2(1)$	$\lambda_5 p_1 \wedge p_2$

Table 3.2.2. Decomposition of a curvature tensor $R \in \mathcal{R}(\mathfrak{g}^{m,\mathfrak{u}})$

$u\wedge v \diagdown F$	$\mathbb{R}(p_1 \wedge q_1 + p_2 \wedge q_2)$	$\mathbb{R}(p_1 \wedge q_2 - p_2 \wedge q_1)$	$\mathbb{R}J_{m+1,\dots,n}$	$\mathfrak{u}$	$\{p_1 \wedge w + p_2 \wedge Jw \mid w \in E^1_{1,\dots,m}\}$	$\{p_1 \wedge Jw - p_2 \wedge w \mid w \in E^1_{1,\dots,m}\}$	$\{p_1 \wedge w + p_2 \wedge Jw \mid w \in E^1_{m+1,\dots,n}\}$	$\mathbb{R}p_1 \wedge p_2$
$p_1 \wedge q_1$	$\lambda_1(p_1 \wedge q_1 + p_2 \wedge q_2)$	$\lambda_2(p_1 \wedge q_2 - p_2 \wedge q_1)$	0	0	$p_1 \wedge N_1(1) + p_2 \wedge JN_1(1)$	$-\big(p_1 \wedge JN_2(1) - p_2 \wedge N_2(1)\big)$	0	$\lambda_4 p_1 \wedge p_2$
$p_1 \wedge q_2$	$-\lambda_2(p_1 \wedge q_1 + p_2 \wedge q_2)$	$\lambda_1(p_1 \wedge q_2 - p_2 \wedge q_1)$	0	0	$p_1 \wedge N_2(1) + p_2 \wedge JN_2(1)$	$-\big(p_1 \wedge JN_1(1) - p_2 \wedge N_1(1)\big)$	0	$\lambda_3 p_1 \wedge p_2$
$x_1 \wedge y_1$	0	0	0	$R_0(x_1 \wedge y_1)$	$\frac{1}{2}\big(p_1 \wedge \mathrm{pr}_{E^1} P^*(x_1 \wedge y_1) + p_2 \wedge J\,\mathrm{pr}_{E^1} P^*(x_1 \wedge y_1)\big)$	$\frac{1}{2}\big(p_1 \wedge J\,\mathrm{pr}_{E^1} JP^*(x_1 \wedge y_1) - p_2 \wedge \mathrm{pr}_{E^1} JP^*(x_1 \wedge y_1)\big)$	0	$\big(\eta((S - S^*)x_1, y_1) + \eta(J_{m,\dots,n} x_1, y_1)\big) \cdot p_1 \wedge p_2$
$q_1 \wedge x$	$-N_1^*(x)(p_1 \wedge q_1 + p_2 \wedge q_2)$	$N_2^*(x)(p_1 \wedge q_2 - p_2 \wedge q_1)$	0	$P(x)$	$p_1 \wedge (-JS^{21}(x)) + p_2 \wedge S^{21}(x)$	$-(p_1 \wedge JS^{11}(x) - p_2 \wedge S^{11}(x))$	0	$M_1^*(x) p_1 \wedge p_2$
$q_2 \wedge x$	$-N_2^*(x)(p_1 \wedge q_1 + p_2 \wedge q_2)$	$-N_1^*(x)(p_1 \wedge q_2 - p_2 \wedge q_1)$	0	$-P(Jx)$	$p_1 \wedge S^{11*}(x) + p_2 \wedge JS^{11*}(x)$	$p_1 \wedge JS^{12}(Jx) - p_2 \wedge S^{12}(Jx))$	0	$M_2^*(x) p_1 \wedge p_2$
$q_1 \wedge y$	0	0	0	0	0	0	$p_1 \wedge \lambda_3 y + p_2 \wedge \lambda_3 Jy$	$M_3^*(y) p_1 \wedge p_2$
$q_1 \wedge q_2$	$-\lambda_4(p_1 \wedge q_1 + p_2 \wedge q_2)$	$\lambda_3(p_1 \wedge q_2 - p_2 \wedge q_1)$	$\lambda_3 J_{m+1,\dots,n}$	$S - S^*$	$p_1 \wedge M_1(1) + p_2 \wedge JM_1(1)$	$p_1 \wedge JM_2(1) - p_2 \wedge M_2(1)$	$p_1 \wedge M_3(1) + p_2 \wedge JM_3(1)$	$\lambda_5 p_1 \wedge p_2$

0. Furthermore, $R_0 \in \mathcal{R}(\mathfrak{u}\cap\mathfrak{u}(m))$, $P \in \mathcal{P}(\mathfrak{u}\cap\mathfrak{u}(m))$, $S^{11} \in \mathrm{Hom}(E_{1,\dots,m}, E^1_{1,\dots,m})$, $S^{21} \in \mathrm{Hom}(E_{1,\dots,m}, E^2_{1,\dots,m})$, $S^{12} \in \mathrm{Hom}(E_{1,\dots,m}, E^1_{1,\dots,m})$, $S^{11}|_{E^2_{1,\dots,m}} = 0$, $S^{21}|_{E^2_{1,\dots,m}} = 0$, $S^{12}|_{E^1_{1,\dots,m}} = 0$, $S^{12*} = JS^{12}J$, $S^{21*} = JS^{21}J$, $S \in \mathrm{Hom}(E_{1,\dots,m}, E_{1,\dots,m})$, $S = S^{11}+S^{12}+S^{21}+JS^{11*}J$ and $S - S^* \in \mathfrak{u}\cap\mathfrak{u}(m)$. Conversely, for any elements as above the linear map $R \in \mathrm{Hom}(\mathbb{R}^{2,2n+2}\wedge\mathbb{R}^{2,2n+2}, \mathfrak{g}^{m,\mathfrak{u}})$ defined by the Table 3.2.2, (14) and (15) satisfies $R \in \mathcal{R}(\mathfrak{g}^{m,\mathfrak{u}})$.

1) Consider the case $n = 0$.

Lemma 3.1. *The Lie algebras of Part 1 of Theorem 3.1 exhaust all weakly-irreducible Berger subalgebras of* $\mathfrak{u}(1,1)_{<p_1,p_2>}$.

Proof. Let $R \in \mathcal{R}(\mathfrak{u}(1,1)_{<p_1,p_2>})$. As above, R can be found from the conditions $R(p_2\wedge q_2) = R(p_1\wedge q_1)$, $R(p_2\wedge q_1) = -R(p_1\wedge q_2)$, $R(p_1\wedge p_2) = 0$ and the following table:

$u\wedge v$ \ F	$\mathbb{R}(p_1\wedge q_1+p_2\wedge q_2)$	$\mathbb{R}(p_1\wedge q_2-p_2\wedge q_1)$	$\mathbb{R}p_1\wedge p_2$
$p_1\wedge q_1$	$\lambda_1(p_1\wedge q_1+p_2\wedge q_2)$	$\lambda_2(p_1\wedge q_2-p_2\wedge q_1)$	$\lambda_4 p_1\wedge p_2$
$p_1\wedge q_2$	$-\lambda_2(p_1\wedge q_1+p_2\wedge q_2)$	$\lambda_1(p_1\wedge q_2-p_2\wedge q_1)$	$\lambda_3 p_1\wedge p_2$
$q_1\wedge q_2$	$-\lambda_4(p_1\wedge q_1+p_2\wedge q_2)$	$\lambda_3(p_1\wedge q_2-p_2\wedge q_1)$	$\lambda_5 p_1\wedge p_2$

We will consider all subalgebras of $\mathfrak{u}(1,1)_{<p_1,p_2>}$ and check which of these subalgebras are weakly-irreducible Berger subalgebras. Let $\mathfrak{g}\subset\mathfrak{u}(1,1)_{<p_1,p_2>}$ be a subalgebra. We have the following cases:

Case 1. $\mathrm{pr}_{\mathcal{C}}\,\mathfrak{g} = \{0\}$, i.e. $\mathfrak{g}\subset\mathcal{A}^1\oplus\mathcal{A}^2$;

Case 2. $\mathcal{C}\subset\mathfrak{g}$;

Case 3. $\mathcal{C}\not\subset\mathfrak{g}$ and $\mathrm{pr}_{\mathcal{C}}\,\mathfrak{g}\neq\{0\}$.

Consider these cases.

Case 1. $\mathrm{pr}_{\mathcal{C}}\,\mathfrak{g} = \{0\}$, i.e. $\mathfrak{g}\subset\mathcal{A}^1\oplus\mathcal{A}^2$. We have the following subcases.

Subcase 1.1. $\mathfrak{g} = \mathcal{A}^1\oplus\mathcal{A}^2$. We claim that $\mathfrak{g}$ is a weakly-irreducible Berger subalgebra. Suppose that $\mathfrak{g}$ preserves a non-trivial vector subspace $L\subset\mathbb{R}^{2,2}$. Let $\alpha_1p_1+\alpha_2p_2+\beta_1q_1+\beta_2q_2\in L$ be a non-zero vector. Applying the element $(1,0)\in\mathcal{A}^1$, we get $\alpha_1p_1+\alpha_2p_2-\beta_1q_1-\beta_2q_2\in L$. Hence, $\alpha_1p_1+\alpha_2p_2\in L$ and $\beta_1q_1+\beta_2q_2\in L$. Applying to these vectors the element $(0,1)\in\mathcal{A}^2$, we get $\alpha_1p_2-\alpha_2p_1\in L$ and $\beta_1q_2-\beta_2q_1\in L$. There are three possibilities: $L=\mathbb{R}^{2,2}$, $L=\mathbb{R}p_1\oplus\mathbb{R}p_2$ or $L=\mathbb{R}q_1\oplus\mathbb{R}q_2$. Hence the subalgebra $\mathfrak{g}$ is weakly-irreducible.

Furthermore, $\mathfrak{g}$ is spanned be the image of the curvature tensor $R\in\mathcal{R}(\mathfrak{g})$ given by $\lambda_1 = 1$ and $\lambda_2=\dots=\lambda_5=0$.

Subcase 1.2. $\mathfrak{g} = \{(a\gamma_1, a\gamma_2, 0)|a \in \mathbb{R}\}$, where $\gamma_1, \gamma_2 \in \mathbb{R}$. We claim that this subalgebra is not a weakly-irreducible Berger subalgebra. Indeed, if $\gamma_2 = 0$, then $\mathfrak{g}$ preserves the non-degenerate proper subspace $\mathbb{R}p_1 \oplus \mathbb{R}q_1 \subset \mathbb{R}^{2,2}$. If $\gamma_1 = 0$, then $\mathfrak{g}$ preserves the non-degenerate proper subspace $\mathbb{R}(p_1 + q_1) \oplus \mathbb{R}(p_2 + q_2) \subset \mathbb{R}^{2,2}$. Suppose that $\gamma_1 \neq 0$ and $\gamma_2 \neq 0$. Let $R \in \mathcal{R}(\mathfrak{g})$. We have

$$R(p_1 \wedge q_1) = \lambda_1(p_1 \wedge q_1 + p_2 \wedge q_2) + \lambda_2(p_1 \wedge q_2 - p_2 \wedge q_1) + \lambda_4(p_1 \wedge p_2),$$

$$R(p_1 \wedge q_2) = -\lambda_2(p_1 \wedge q_1 + p_2 \wedge q_2) + \lambda_1(p_1 \wedge q_2 - p_2 \wedge q_1) + \lambda_3(p_1 \wedge p_2).$$

Hence, $\lambda_3 = \lambda_4 = 0$ and $\frac{\lambda_1}{-\lambda_2} = \frac{\lambda_2}{\lambda_1}$. Therefore, $\lambda_1 = \lambda_2 = 0$. Moreover, $R(q_1 \wedge q_2) = \lambda_5(p_1 \wedge p_2)$. Consequently, $\lambda_5 = 0$. Thus, $R = 0$, $\mathcal{R}(\mathfrak{g}) = \{0\}$ and $\mathfrak{g}$ is not a Berger algebra.

Case 2. $\mathcal{C} \subset \mathfrak{g}$. We have the following subcases:

Subcase 2.1. $\mathfrak{g} = (\mathcal{A}^1 \oplus \mathcal{A}^2) \ltimes \mathcal{C} = \mathfrak{u}(1,1)_{<p_1,p_2>}$;

Subcase 2.2. $\mathfrak{g} = \{(a\gamma_1, a\gamma_2, 0)|a \in \mathbb{R}\} \ltimes C$, where $\gamma_1, \gamma_2 \in \mathbb{R}$. These subalgebras contain $\mathcal{C}$, hence they are weakly-irreducible (Part 1 of Theorem 2.1). These subalgebras are Berger algebras, since any element of these algebras can be obtained as $R(q_1 \wedge q_2)$ for some curvature tensor R.

Case 3. $\mathcal{C} \not\subset \mathfrak{g}$ and $\mathrm{pr}_{\mathcal{C}}\, \mathfrak{g} \neq \{0\}$. Consider the following subcases.

Subcase 3.1. $\dim \mathfrak{g} = 1$, then $\mathfrak{g} = \{c\gamma_1, c\gamma_2, c)|c \in \mathbb{R}\}$, where $\gamma_1, \gamma_2 \in \mathbb{R}$, $\gamma_1 \neq 0$ or $\gamma_2 \neq 0$. We claim that $\mathfrak{g}$ is not a Berger algebra. Indeed, let $R \in \mathcal{R}(\mathfrak{g})$ by analogy with Subcase 1.2, we have $\frac{\lambda_1}{-\lambda_2} = \frac{\lambda_2}{\lambda_1} = \frac{\lambda_4}{\lambda_3}$. Hence, $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = \lambda_4 = \lambda_5 = 0$.

Subcase 3.2. $\dim \mathfrak{g} = 2$, then $\mathrm{pr}_{\mathcal{A}^1 \oplus \mathcal{A}^2}\, \mathfrak{g} = \mathcal{A}^1 \oplus \mathcal{A}^2$ and $\mathfrak{g} = \{(a_1, a_2, \nu(a_1, a_2))|a_1, a_2 \in \mathbb{R}\}$, where $\nu : \mathbb{R} \oplus \mathbb{R} \to \mathbb{R}$ is a non-zero linear map. Let $\gamma_1, \gamma_2 \in \mathbb{R}$ be numbers such that $\nu(\gamma_1, \gamma_2) = 0$ and $\nu(-\gamma_2, \gamma_1) = 1$. Hence $\mathfrak{g}$ has the form $\{(a\gamma_1, a\gamma_2, 0)|a \in \mathbb{R}\} \ltimes \{-c\gamma_2, c\gamma_2, c)|c \in \mathbb{R}\}$. We have $[(\gamma_1, \gamma_2, 0), (-\gamma_2, \gamma_1, 1)] = (0, 0, 2\gamma_1) \in \mathfrak{g}$. Therefore, $\gamma_1 = 0$. Let $\gamma = \gamma_2$. Thus, $\mathfrak{g} = \mathcal{A}^2 \oplus \{(c\gamma, 0, c)|c \in \mathbb{R}\}$, $\gamma \neq 0$. This Lie algebra is conjugated to the Lie algebra $\mathfrak{hol}^2_{n=0}$. To see this it is enough to choose the new basis $p_1, p_2, q_1 - \frac{1}{2\gamma}p_2, q_2 + \frac{1}{2\gamma}p_1$.

The lemma is proved. $\Box$

2) Let $n \geq 1$. We claim that the subalgebras of $\mathfrak{u}(1, n+1)_{<p_1,p_2>}$ from the statement of the theorem are weakly-irreducible Berger algebras. Indeed, these subalgebras are weakly-irreducible. Table 3.2.2 shows that all these subalgebras are Berger algebras (any element of each algebra can be obtained as $R(q_1 \wedge q_2)$ for proper curvature tensor R), this will follow also from Theorem 3.2.

We must prove that there are no other weakly-irreducible Berger subalgebras of $\mathfrak{u}(1,n+1)_{<p_1,p_2>}$. For this we need all candidates for the weakly-irreducible subalgebras of $\mathfrak{u}(1,n+1)_{<p_1,p_2>}$. Recall that in order to classify weakly-irreducible subalgebras of $\mathfrak{su}(1,n+1)_{<p_1,p_2>}$ we considered a Lie algebras homomorphism $\Gamma : \mathfrak{u}(1,n+1)_{<p_1,p_2>} \to \mathrm{LA}(\mathrm{Sim}\,\mathcal{H}_n)$ and its restriction $\Gamma_{\mathfrak{su}(1,n+1)_{<p_1,p_2>}} : \mathfrak{su}(1,n+1)_{<p_1,p_2>} \to \mathrm{LA}(\mathrm{Sim}\,\mathcal{H}_n)$ which is an isomorphism. We proved that if $\mathfrak{g} \subset \mathfrak{u}(1,n+1)_{<p_1,p_2>}$ is weakly-irreducible, then the subalgebra $\mathfrak{f} = \Gamma(\mathfrak{g}) \subset \mathrm{LA}(\mathrm{Sim}\,\mathcal{H}_n)$ satisfies a property. Then we found all subalgebras $\mathfrak{f} \subset \mathrm{LA}(\mathrm{Sim}\,\mathcal{H}_n)$ that satisfy this property and the isomorphism $\Gamma_{\mathfrak{su}(1,n+1)_{<p_1,p_2>}}$ gave us the list of candidates for the weakly-irreducible subalgebras of $\mathfrak{su}(1,n+1)_{<p_1,p_2>}$. Now, since $\ker \Gamma_{\mathfrak{su}(1,n+1)_{<p_1,p_2>}} = \mathbb{R}J$, any weakly-irreducible subalgebra of $\mathfrak{u}(1,n+1)_{<p_1,p_2>}$ must have one of the forms $\mathfrak{g}$, $\mathfrak{g}\oplus\mathbb{R}J$ or $\mathfrak{g}^\xi$, where $\mathfrak{g}$ is a candidate to the weakly-irreducible subalgebras of $\mathfrak{su}(1,n+1)_{<p_1,p_2>}$, $\mathfrak{g}^\xi = \{x+\xi(x)J | x \in \mathfrak{g}\}$ and $\xi : \mathfrak{g} \to \mathbb{R}$ is a non-zero linear map such that $\mathfrak{g}^\xi$ is a Lie algebra. Recall that for $m > 0$ all candidates to the weakly-irreducible subalgebras of $\mathfrak{su}(1,n+1)_{<p_1,p_2>}$ are weakly-irreducible subalgebras.

Lemma 3.2. *Let $\mathfrak{g}$ be one of the following candidates to the weakly-irreducible subalgebra of $\mathfrak{su}(1,n+1)_{<p_1,p_2>}$:*

$$\mathfrak{g}^{0,\mathfrak{h},\psi,k},\quad \mathfrak{g}^{0,\mathfrak{h},\mathcal{A}^1,\zeta},\quad \mathfrak{g}^{0,\mathfrak{h},\varphi,i_0,\zeta},\quad \mathfrak{g}^{0,\mathfrak{h},\psi,k,i_0,\zeta}$$

(these Lie algebras defined as in the proof of Theorem 2.1, see Lemmas 2.2, 2.3 and 2.4). Then the Lie algebras of the form $\mathfrak{g}$, $\mathfrak{g}^\xi$ and $\mathfrak{g}\oplus\mathbb{R}J$ are not Berger algebras.

Proof. Let $\mathfrak{g}$ be one of the above Lie algebras and $R \in \mathcal{R}(\mathfrak{g}\oplus\mathbb{R}J)$. Since $\mathfrak{h} \subset \mathfrak{sod}(1,...,n)$, from Table 3.2.2 it follows that $(\mathrm{pr}_{u(n)} R(\mathbb{R}^{2,2n+2} \wedge \mathbb{R}^{2,2n+2})) \cap \mathfrak{h} = \{0\}$. Therefore, if one of the Lie algebras $\mathfrak{g}$, $\mathfrak{g}^\xi$ and $\mathfrak{g}\oplus\mathbb{R}J$ is a Berger algebra, then $\mathfrak{h} = \{0\}$. This condition holds only for the Lie algebra $\mathfrak{g} = \mathfrak{g}^{0,\mathfrak{h}=\{0\},\mathcal{A}^1,\zeta}$. We claim that $\mathcal{R}(\mathfrak{g}\oplus\mathbb{R}J) = \{0\}$. Indeed, let $R \in \mathcal{R}(\mathfrak{g}\oplus\mathbb{R}J)$. We decompose R as in Table 3.2.2. For any $y \in E^1_{1,\dots,m}$ we have $R(q_1\wedge y) = p_1 \wedge \lambda_3 y + p_2 \wedge \lambda_3 Jy + M_3^*(y) p_1 \wedge p_2 \in \mathfrak{g}$. Consequently, $\lambda_3 = 0$ and $M_3 = 0$. It is easy to show in the same way that all the other components of R are also zero. Thus the Lie algebras of the form $\mathfrak{g}$, $\mathfrak{g}^\xi$ and $\mathfrak{g}\oplus\mathbb{R}J$ are not Berger algebras. The lemma is proved. $\square$

Lemma 3.3. *Let $\mathfrak{g} \subset \mathfrak{u}(1,n+1)_{<p_1,p_2>}$ be a weakly-irreducible Berger subalgebra with the associated number m, $1 \leq m < n$. If $\mathfrak{g}$ does not contain the set $\{p_1\wedge w + p_2 \wedge Jw | w \in E^1_{m+1,\dots,n}\}$, then $\mathrm{pr}_{\mathfrak{u}(m+1,\dots,n)}\,\mathfrak{g} = \{0\}$.*

Proof. Consider a curvature tensor $R \in \mathcal{R}(\mathfrak{g})$. We decompose R using Table 3.2.2. Since $m \geq 1$, we see that $p_1 \wedge p_2 \in \mathfrak{g}$. Let $y \in E^1_{m+1,\dots,n}$ be a vector such that $p_1 \wedge y + p_2 \wedge Jy \notin \mathfrak{g}$, then $R(q_1 \wedge y) = p_1 \wedge \lambda_3 y + p_2 \wedge \lambda_3 Jy + M_3^*(y) p_1 \wedge p_2 \in \mathfrak{g}$. Consequently, $\lambda_3 = 0$. From Table 3.2.2 it follows that $\operatorname{pr}_{\mathfrak{u}(m+1,\dots,n)} R(\mathbb{R}^{2,2n+2} \wedge \mathbb{R}^{2,2n+2}) = \{0\}$. This proves the lemma. $\Box$

Lemma 3.4. *Let $\mathfrak{g}$ be a Lie algebra of the form $\mathfrak{g}^{n,\mathfrak{h},\psi,k,l}$ or $\mathfrak{g}^{m,\mathfrak{h},\psi,k,l,r}$. Then*

1) *If $\mathfrak{g}$ is a Berger algebra, then $\mathfrak{h} \subset \mathfrak{su}(k)$.*

2) *If $\xi : \mathfrak{g} \to \mathbb{R}$ is not zero and $\mathfrak{g}^\xi$ is a Berger algebra, then there exist elements $A = \left(\begin{smallmatrix} B & -C \\ C & B \end{smallmatrix}\right) \in \mathfrak{u}(n)$, $z_1, z_2 \in \mathbb{R}^n$ such that $A + J_k - \frac{k}{n+2} J_n \in \mathfrak{z}(\mathfrak{h})$, for $x = (0, -\frac{k}{n+2}, B, C + J_k - \frac{k}{n+2} J_n, z_1, z_2, 0) \in \mathfrak{g}$ we have $\xi(x) = \frac{k}{n+2}$, ξ is zero on the orthogonal complement to $\mathbb{R}x$, and the orthogonal complement to $\mathbb{R}(A + J_k - \frac{k}{n+2} J_n)$ in $\mathfrak{u}(n)$ is contained in $\mathfrak{u}(k)$. In particular, $x + \xi(x) J = (0, 0, B, C + J_k, z_1, z_2, 0)$ and $\operatorname{pr}_{\mathfrak{u}(n)} \mathfrak{g}^\xi \subset \mathfrak{u}(k)$.*

3) *The Lie algebra $\mathfrak{g} \oplus \mathbb{R}J$ is not a Berger algebra.*

Proof. Statements 1) and 3) follow from Lemma 3.3. Let us prove Statement 2). Suppose that $\mathfrak{g}^\xi$ is a Berger algebra for some linear map $\xi : \mathfrak{g} \to \mathbb{R}$. From Lemma 3.3 it follows that $\operatorname{pr}_{\mathfrak{u}(k+1,\dots,n)} \mathfrak{g}^\xi = \{0\}$. This can happen only in the case described in Statement 2). In this situation the Lie algebra $\mathfrak{g}^\xi$ is either of the form $\mathfrak{hol}^{n,\mathfrak{u},\psi_1,k_1,l_1}$ or $\mathfrak{g}^{m,\mathfrak{u},\psi_1,k_1,l_1,r_1}$. The lemma is proved. $\Box$

Now we have to consider only the Lie algebras of the form $\mathfrak{g}$, $\mathfrak{g}^\xi$ and $\mathfrak{g} \oplus \mathbb{R}J$ for $\mathfrak{g} = \mathfrak{g}^{m,\mathfrak{h},\mathcal{A}_1}$ and $\mathfrak{g} = \mathfrak{g}^{m,\mathfrak{h},\varphi}$, where $0 \leq m \leq n$ and $\mathfrak{h} \subset \mathfrak{su}(m) \oplus \mathbb{R}(J_m - \frac{m}{n+2} J_n) \oplus \mathfrak{sod}(m+1, ..., n)$. Lemma 3.3 yields that if any of these Lie algebras is a Berger algebra, then $\mathfrak{h} \subset \mathfrak{su}(m) \oplus \mathbb{R}(J_m - \frac{m}{n+2} J_n)$.

Let us consider the Lie algebra $\mathfrak{g} = \mathfrak{g}^{m,\mathfrak{h},\mathcal{A}_1}$, where $\mathfrak{h} \subset \mathfrak{su}(m) \oplus \mathbb{R}(J_m - \frac{m}{n+2} J_n)$. Obviously, $\mathfrak{g}$ is a holonomy algebra of the form $\mathfrak{hol}^{m,\mathfrak{u},\mathcal{A}^1,\phi}$, where $\mathfrak{u} = \operatorname{pr}_{\mathfrak{u}(m)} \mathfrak{h}$ and $\phi : \mathfrak{u} \to \mathbb{R}$ is the map linear map given by $\phi : \left(\begin{smallmatrix} B & -C \\ C & B \end{smallmatrix}\right) \mapsto -\frac{1}{2} \operatorname{tr} C$. Consider a Lie algebra of the form $\mathfrak{g}^\xi$, where $\xi : \mathfrak{g} \to \mathbb{R}$ is a non-zero linear map. We have $\xi|_{\mathfrak{g}'} = \xi|_{\mathfrak{h}' \ltimes (\mathcal{N}^1 + \mathcal{N}^{1,\dots,m} + \mathcal{C})} = 0$, i.e. ξ can be considered as a linear map $\xi : \mathcal{A}^1 \oplus \mathfrak{z}(\mathfrak{h}) \to \mathbb{R}$. If $\xi|_{\mathcal{A}^1} \neq 0$ and $\xi|_{\mathfrak{z}(\mathfrak{h})} = 0$, then $\mathfrak{g}^\xi$ is a holonomy algebra of the form $\mathfrak{hol}^{m,\mathfrak{u},\varphi,\phi}$ or $\mathfrak{hol}^{m,\mathfrak{u},\lambda}$. Suppose that $\xi|_{\mathcal{A}^1} = 0$ and let $A \in \mathfrak{h}$ be an element such that $\xi(A) \neq 0$ and ξ is zero on the orthogonal complement to $\mathbb{R}A$ in $\mathfrak{h}$. Consider the decomposition $A = A_1 + a(J_m - \frac{m}{n+2} J_n)$, where $A_1 \in \mathfrak{su}(m)$ and $a \in \mathbb{R}$. If $A_1 \neq 0$, then $\mathfrak{g}^\xi$ is a holonomy algebra of the form $\mathfrak{hol}^{m,\mathfrak{u},\mathcal{A}^1,\phi}$. Suppose that $A_1 = 0$. If

$\xi(J_m - \frac{m}{n+2}J_n) = -J$, then $\mathfrak{g}^\xi$ is a holonomy algebra of the form $\mathfrak{hol}^{m,\mathfrak{u},\mathcal{A}^1,\tilde{A}^2}$, otherwise, then $\mathfrak{g}^\xi$ is a holonomy algebra of the form $\mathfrak{hol}^{m,\mathfrak{u},\mathcal{A}^1,\phi}$. Obviously, the Lie algebra $\mathfrak{g}\oplus\mathbb{R}J$ is a holonomy algebra of the form $\mathfrak{hol}^{m,\mathfrak{u},\mathcal{A}^1,\tilde{A}_2}$ or $\mathfrak{hol}^{m,\mathfrak{u},\mathcal{A}^1,\phi}$.

The case $\mathfrak{g} = \mathfrak{g}^{m,\mathfrak{h},\varphi}$ can be considered in the same way.

Thus the Lie algebras of the theorem exhaust weakly-irreducible Berger subalgebras of $\mathfrak{u}(1,n+1)_{<p_1,p_2>}$. The proof of the theorem will follow from Theorem 3.2. $\Box$

3.3 Proof of Theorem 3.2

Since the coefficients of each our metric g are polynomial functions, the Levi-Civita connection given by g is analytic and the Lie algebra $\mathfrak{hol}_0$ is generated by the operators

$$R(X,Y)_0, \nabla R(X,Y;Z_1)_0, \nabla^2 R(X,Y;Z_1;Z_2)_0, ... \in \mathfrak{so}(T_0\mathbb{R}^{2n+4}, g_0),$$

where $\nabla^r R(X,Y;Z_1;...;Z_r) = (\nabla_{Z_r}\cdots\nabla_{Z_1}R)(X,Y)$ and X, Y, Z_1, Z_2,... are vectors at the point 0.

We consider the case $n>0$. The proof for the case $n=0$ can be obtained by simple computations.

First we consider some general metric and find all covariant derivatives of the curvature tensor of this metric. Let $1\leq n_0\leq m\leq n$ be integers as in Section 3.1. We will use the following convention about the ranks of the indices

$$\begin{array}{lll} a,b,c,d = 1,...,2n+4, & i,j = 3,...,2n+2, & \\ \hat{i},\hat{j} = 3,...,n_0+2, & \tilde{i},\tilde{j} = n_0+3,...,m+2, & \breve{i},\breve{j} = m+3,...,n+2, \end{array}$$

$$\begin{aligned} &\hat{\hat{i}},\hat{\hat{j}} = 3,...,n_0+2, n+3,...,n+n_0+3,\\ &\tilde{\tilde{i}},\tilde{\tilde{j}} = n_0+3,...,m+2, n+n_0+3,...,n+m+2,\\ &\breve{\breve{i}},\breve{\breve{j}} = m+3,...,n+2, n+m+3,...,2n+2. \end{aligned}$$

We will use the Einstein rule for sums.

We assume that the numbers $B^{\hat{i}}_{\alpha\hat{j}}$ and $C^{\hat{i}}_{\alpha\hat{j}}$ equal $B^{\hat{i}-2}_{\alpha\hat{j}-2}$ and $C^{\hat{i}-2}_{\alpha\hat{j}-2}$, respectively (here $(B^i_{\alpha j})^{n_0}_{i,j=1}$ and $(C^i_{\alpha j})^{n_0}_{i,j=1}$ are numbers as in Section 3.1). Define the numbers $A^i_{\alpha j}$ such that $A^{\hat{i}}_{\alpha\hat{j}} = B^{\hat{i}}_{\alpha\hat{j}}$, $A^{\hat{i}+n}_{\alpha\hat{j}+n} = B^{\hat{i}}_{\alpha\hat{j}}$, $A^{\hat{i}+n}_{\alpha\hat{j}} = C^{\hat{i}}_{\alpha\hat{j}}$, $A^{\hat{i}}_{\alpha\hat{j}+n} = -C^{\hat{i}}_{\alpha\hat{j}}$, and $A^i_{\alpha j}=0$ for other i and j, here $\alpha = 1,...,N$.

Let $\varphi,\phi:\mathfrak{u}\to\mathbb{R}$ be two linear maps with $\varphi|_{\mathfrak{u}'} = \phi|_{\mathfrak{u}'} = 0$. Let the numbers φ_α and ϕ_α $(N_1+1\leq\alpha\leq N)$ be as in Section 3.1. If $\varphi=\phi=0$, then we set $N_0 = N+2$ and consider

some numbers φ_{N+1}, φ_{N+2}, ϕ_{N+1}, ϕ_{N+2}. If $\varphi=0$ and $\phi\neq 0$, then we set $N_0=N+1$, $\phi_{N+1}=0$ and consider a number φ_{N+1}. If $\varphi\neq 0$ and $\phi=0$, then we set $N_0=N+1$, $\varphi_{N+1}=0$ and consider a number ϕ_{N+1}. If $\varphi\neq 0$ and $\phi\neq 0$, then we set $N_0=N$. Thus we get some numbers $(\varphi_\alpha)_{\alpha=N_1+1}^{N_0}$ and $(\phi_\alpha)_{\alpha=N_1+1}^{N_0}$. Consider the following polynomials

$$\hat\varphi(x^{2n+3})=\sum_{\alpha=N_1+1}^{N_0}\frac{1}{\alpha!}\varphi_\alpha(x^{2n+3})^\alpha \text{ and } \hat\phi(x^{2n+3})=\sum_{\alpha=N_1+1}^{N_0}\frac{1}{\alpha!}\phi_\alpha(x^{2n+3})^\alpha.$$

Consider the metric g given by (1) with the functions

$$f_i=f_i^\varphi+f_i^\phi+\hat f_i(x^{\hat{\hat{i}}},x^{2n+3})+\tilde f^m_{i\,n_0+1}+\breve f^n_{i\,N_0+1\;m+1}\qquad (i=1,2,3),$$

where $\tilde f^m_{i\,n_0+1}$ and $\breve f^n_{i\,N_0+1\;m+1}$ are functions as in Section 3.1, f_i^φ and f_i^ϕ are functions defined as in Section 3.1 using N_0 instead of N, and $\hat f_i(x^{\hat{\hat{i}}},x^{2n+3})$ are some functions, $\hat f_3(x^{\hat{\hat{i}}},x^{2n+3})=0$.

We assume that $f_1(0)=f_2(0)=f_3(0)=0$, then $g_0=\eta$ and we can identify the tangent space to $\mathbb{R}^{2n+4}$ at 0 with the vector space $\mathbb{R}^{2,2n+2}$ such that $\frac{\partial}{\partial x^1}|_0=p_1$, $\frac{\partial}{\partial x^2}|_0=p_2$, $\frac{\partial}{\partial x^3}|_0=e_1$,...,$\frac{\partial}{\partial x^{n+2}}|_0=e_n$, $\frac{\partial}{\partial x^{n+3}}|_0=f_1$,...,$\frac{\partial}{\partial x^{2n+2}}|_0=f_n$, $\frac{\partial}{\partial x^{2n+3}}|_0=q_1$, $\frac{\partial}{\partial x^{2n+4}}|_0=q_2$. For the non-zero Christoffel symbols of the metric g we have

$$\Gamma^1_{1\,2n+3}=\frac12\frac{\partial f_1}{\partial x^1},\quad \Gamma^1_{1\,2n+4}=\frac12\frac{\partial f_3}{\partial x^1},\quad \Gamma^1_{2\,2n+3}=\frac12\frac{\partial f_1}{\partial x^2},\quad \Gamma^1_{2\,2n+4}=\frac12\frac{\partial f_3}{\partial x^2},\quad \Gamma^1_{i\,2n+3}=\frac12\frac{\partial f_1}{\partial x^i},\tag{16}$$

$$\Gamma^1_{i\,2n+4}=\frac12\left(-\frac{\partial u^i}{\partial x^{2n+3}}+\frac{\partial f_3}{\partial x^i}\right),\quad \Gamma^1_{2n+3\,2n+3}=\frac12\left(\frac{\partial f_1}{\partial x^{2n+3}}+f_3\frac{\partial f_1}{\partial x^2}+f_1\frac{\partial f_1}{\partial x^1}\right),\tag{17}$$

$$\Gamma^1_{2n+3\,2n+4}=\frac12\left(f_3\frac{\partial f_3}{\partial x^2}+f_1\frac{\partial f_3}{\partial x^1}\right),\quad \Gamma^1_{2n+4\,2n+4}=\frac12\left(-\frac{\partial f_2}{\partial x^{2n+3}}+f_3\frac{\partial f_2}{\partial x^2}+f_1\frac{\partial f_2}{\partial x^1}\right),\tag{18}$$

$$\Gamma^2_{1\,2n+3}=\frac12\frac{\partial f_3}{\partial x^1},\quad \Gamma^2_{1\,2n+4}=\frac12\frac{\partial f_2}{\partial x^1},\quad \Gamma^2_{2\,2n+3}=\frac12\frac{\partial f_3}{\partial x^2},\quad \Gamma^2_{2\,2n+4}=\frac12\frac{\partial f_2}{\partial x^2},\quad \Gamma^2_{i\,j}=\frac12\left(\frac{\partial u^i}{\partial x^j}+\frac{\partial u^j}{\partial x^i}\right),\tag{19}$$

$$\Gamma^2_{i\,2n+3}=\frac12\left(\frac{\partial u^i}{\partial x^{2n+3}}+\frac{\partial f_3}{\partial x^i}\right),\quad \Gamma^1_{i\,2n+4}=\frac12\left(\sum_{j=3}^{2n+2}u^j\left(\frac{\partial u^i}{\partial x^j}-\frac{\partial u^j}{\partial x^i}\right)+\frac{\partial f_2}{\partial x^i}\right),\tag{20}$$

$$\Gamma^2_{2n+3\,2n+3}=\frac12\left(2\frac{\partial f_3}{\partial x^{2n+3}}+\sum_{i=3}^{2n+2}u^i\frac{\partial f_1}{\partial x^i}+\left(f_2-\sum_{i=3}^{2n+2}(u^i)^2\right)\frac{\partial f_1}{\partial x^2}+f_3\frac{\partial f_1}{\partial x^1}\right),\tag{21}$$

$$\Gamma^2_{2n+3\,2n+4}=\frac12\left(\sum_{i=3}^{2n+2}u^i\left(-\frac{\partial u^i}{\partial x^{2n+3}}+\frac{\partial f_3}{\partial x^i}\right)+\frac{\partial f_2}{\partial x^{2n+3}}+\left(f_2-\sum_{i=3}^{2n+2}(u^i)^2\right)\frac{\partial f_3}{\partial x^2}+f_3\frac{\partial f_3}{\partial x^1}\right),\tag{22}$$

$$\Gamma^2_{2n+4\,2n+4}=\frac12\left(\sum_{i=3}^{2n+2}u^i\frac{\partial f_2}{\partial x^i}+\left(f_2-\sum_{i=3}^{2n+2}(u^i)^2\right)\frac{\partial f_2}{\partial x^2}+f_3\frac{\partial f_2}{\partial x^1}\right),\quad \Gamma^i_{j\,2n+4}=\frac12\left(\frac{\partial u^i}{\partial x^j}-\frac{\partial u^j}{\partial x^i}\right),\tag{23}$$

$$\Gamma^i_{2n+3\,2n+3}=\frac12\left(-\frac{\partial f_1}{\partial x^i}+u^i\frac{\partial f_1}{\partial x^2}\right),\quad \Gamma^i_{2n+3\,2n+4}=\frac12\left(\frac{\partial u^i}{\partial x^{2n+3}}-\frac{\partial f_3}{\partial x^i}+u^i\frac{\partial f_3}{\partial x^2}\right),\tag{24}$$

$$\Gamma^i_{2n+4\,2n+4}=\frac12\left(-\frac{\partial f_2}{\partial x^i}+u^i\frac{\partial f_2}{\partial x^2}\right),\quad \Gamma^{2n+3}_{2n+3\,2n+3}=-\frac12\frac{\partial f_1}{\partial x^1},\quad \Gamma^{2n+3}_{2n+3\,2n+4}=-\frac12\frac{\partial f_3}{\partial x^1},\tag{25}$$

$$\Gamma^{2n+3}_{2n+4\,2n+4}=-\frac12\frac{\partial f_2}{\partial x^1},\quad \Gamma^{2n+4}_{2n+3\,2n+3}=-\frac12\frac{\partial f_1}{\partial x^2},\quad \Gamma^{2n+4}_{2n+3\,2n+4}=-\frac12\frac{\partial f_3}{\partial x^2},\quad \Gamma^{2n+4}_{2n+4\,2n+4}=-\frac12\frac{\partial f_2}{\partial x^2}.\tag{26}$$

Note that if $a \notin \{1,2\}$, then $\Gamma^a_{1b} = \Gamma^a_{2b} = 0$. This means that the holonomy algebra $\mathfrak{hol}_0$ of the metric g at the point 0 preserves the vector subspace $\mathbb{R}p_1 \oplus \mathbb{R}p_2 \subset \mathbb{R}^{2,2n+2} = T_0\mathbb{R}^{2n+4}$, hence $\mathfrak{hol}_0$ is contained in $\mathfrak{so}(2,2n+2)_{<p_1,p_2>}$, where

$$\mathfrak{so}(2,2n+2)_{<p_1,p_2>} = \left\{ \left(\begin{array}{ccccc} a_{11} & a_{12} & -X^t & 0 & -c \\ a_{21} & a_{22} & -Y^t & c & 0 \\ 0 & 0 & A & X & Y \\ 0 & 0 & 0 & -a_{11} & -a_{21} \\ 0 & 0 & 0 & -a_{12} & -a_{22} \end{array} \right) \middle| \begin{array}{c} \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right) \in \mathfrak{gl}(2), \\ c \in \mathbb{R}, \\ X, Y \in \mathbb{R}^{2n}, \\ A \in \mathfrak{so}(2n) \end{array} \right\}.$$

In particular, it is enough to compute the following components of the covariant derivatives of the curvature tensor: $R^a_{bcd;a_1;a_2;\dots}$, $R^a_{icd;a_1;a_2;\dots}$, $R^i_{jcd;a_1;a_2;\dots}$, where $a,b \in \{1,2\}$ and $3 \le i,j \le 2n+2$. We have

$$R^1_{1\,2n+3\,2n+4} = R^2_{2\,2n+3\,2n+4} = \sum_{\alpha=N_1+1}^{N_0} \frac{1}{(\alpha-1)!}\varphi_\alpha (x^{2n+3})^{\alpha-1},$$
$$R^1_{1ab} = R^2_{2ab} = 0 \text{ if } \{a\}\cup\{b\} \neq \{2n+3, 2n+4\}, \quad (27)$$

$$R^2_{1\,2n+3\,2n+4} = -R^1_{2\,2n+3\,2n+4} = \sum_{\alpha=N_1+1}^{N_0} \frac{1}{(\alpha-1)!}\phi_\alpha (x^{2n+3})^{\alpha-1},$$
$$R^2_{1ab} = R^1_{2ab} = 0 \text{ if } \{a\}\cup\{b\} \neq \{2n+3, 2n+4\}, \quad (28)$$

$$R^1_{ij\,2n+3} = \frac{1}{2}\frac{\partial^2 f_1}{\partial x^i \partial x^j} - \frac{1}{4}\left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i}\right)\frac{\partial f_1}{\partial x^2}, \quad (29)$$
$$R^1_{ij\,2n+4} = -\frac{1}{2}\frac{\partial^2 u^i}{\partial x^j \partial x^{2n+3}} + \frac{1}{2}\frac{\partial^2 f_3}{\partial x^i \partial x^j} - \frac{1}{4}\left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i}\right)\frac{\partial f_3}{\partial x^2}, \quad (30)$$
$$R^2_{ij\,2n+3} = -\frac{1}{2}\frac{\partial^2 u^j}{\partial x^i \partial x^{2n+3}} + \frac{1}{2}\frac{\partial^2 f_3}{\partial x^i \partial x^j} - \frac{1}{4}\left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i}\right)\frac{\partial f_3}{\partial x^2}, \quad (31)$$

$$R^2_{ij\,2n+4} = \frac{1}{4}\sum_{i_1=3}^{2n+2}\left(\frac{\partial u^{i_1}}{\partial x^i} - \frac{\partial u^i}{\partial x^{i_1}}\right)\left(\frac{\partial u^j}{\partial x^{i_1}} - \frac{\partial u^{i_1}}{\partial x^j}\right) + \frac{1}{2}\sum_{i_1=3}^{2n+2} u^{i_1}\left(\frac{\partial^2 u^i}{\partial x^{i_1}\partial x^j} - \frac{\partial^2 u^{i_1}}{\partial x^i \partial x^j}\right)$$
$$+ \frac{1}{2}\frac{\partial^2 f_2}{\partial x^i \partial x^j} - \frac{1}{4}\left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i}\right)\frac{\partial f_2}{\partial x^2}, \quad (32)$$

$$R^1_{i\,2n+3\,2n+4} = \frac{1}{4}\Bigg(-2\frac{\partial^2 u^j}{(\partial x^{2n+3})^2} + 2\frac{\partial^2 f_3}{\partial x^i \partial x^{2n+3}} - \sum_{j=3}^{2n+2}\left(\frac{\partial u^i}{\partial x^j} - \frac{\partial u^j}{\partial x^i}\right)\frac{\partial f_1}{\partial x^j}$$
$$+ \left(\sum_{j=3}^{2n+2} u^j\left(\frac{\partial u^i}{\partial x^j} - \frac{\partial u^j}{\partial x^i}\right) + \frac{\partial f_2}{\partial x^i}\right)\frac{\partial f_1}{\partial x^2}$$
$$- \frac{\partial u^i}{\partial x^{2n+3}}\frac{\partial f_3}{\partial x^2} - \frac{\partial u^i}{\partial x^{2n+3}}\frac{\partial f_1}{\partial x^1} - \frac{\partial f_1}{\partial x^i}\frac{\partial f_3}{\partial x^1} - \frac{\partial f_3}{\partial x^i}\frac{\partial f_3}{\partial x^2} + \frac{\partial f_3}{\partial x^i}\frac{\partial f_1}{\partial x^1}\Bigg), \quad (33)$$

$$R^2_{i\,2n+3\,2n+4} = \frac{1}{4}\Bigg(2\sum_{j=3}^{2n+2} u^j\left(\frac{\partial^2 u^i}{\partial x^j \partial x^{2n+3}} - \frac{\partial^2 u^j}{\partial x^i \partial x^{2n+3}}\right) + \sum_{j=3}^{2n+2}\left(\frac{\partial u^j}{\partial x^{2n+3}} - \frac{\partial f_3}{\partial x^j}\right)\left(\frac{\partial u^i}{\partial x^j} - \frac{\partial u^j}{\partial x^i}\right)$$
$$+ \frac{\partial f_2}{\partial x^i}\frac{\partial f_3}{\partial x^2} + \frac{\partial^2 f_2}{\partial x^i \partial x^{2n+3}} - \frac{\partial u^i}{\partial x^{2n+3}}\frac{\partial f_2}{\partial x^2} - \frac{\partial u^i}{\partial x^{2n+3}}\frac{\partial f_3}{\partial x^1} - \frac{\partial f_1}{\partial x^i}\frac{\partial f_2}{\partial x^1} - \frac{\partial f_3}{\partial x^i}\frac{\partial f_2}{\partial x^2} - \frac{\partial f_3}{\partial x^i}\frac{\partial f_3}{\partial x^1}\Bigg), \quad (34)$$

$$R^i_{j\,2n+3\,2n+4} = \sum_{\alpha=1}^{N} \frac{1}{(\alpha-1)!} A^i_{\alpha j} (x^{2n+3})^{\alpha-1}, \tag{35}$$

$$R^{\breve{i}}_{\breve{\breve{j}}\,2n+3\,2n+4} = -\, R^{\breve{\breve{j}}}_{\breve{i}\,2n+3\,2n+4} = \delta^{\breve{i}+n}_{\breve{\breve{j}}} \sum_{\alpha=N_1+1}^{N_0} \frac{1}{(\alpha-1)!} \phi_\alpha (x^{2n+3})^{\alpha-1}, \quad R^{\breve{i}}_{j\,2n+3\,2n+4} = R^{\breve{\breve{i}}}_{\breve{\breve{j}}\,2n+3\,2n+4} = 0, \tag{36}$$

$$R^i_{jab} = 0 \text{ if } \{a\} \cup \{b\} \neq \{2n+3, 2n+4\}, \tag{37}$$

$$R^i_{j\,2n+3\,2n+4} = 0 \text{ if } i, j \notin \{3, ..., n_0+2, n+3, ..., n+n_0+2\} \text{ or } i, j \notin \{m+3, ..., n+2, n+m+3, ..., 2n+2\}. \tag{38}$$

Specifically,

$$R^1_{\tilde{\tilde{i}}\tilde{j}\,2n+3} = R^2_{n+\tilde{i}\tilde{j}\,2n+3} = \delta_{\tilde{\tilde{i}}\tilde{\tilde{j}}}, \quad R^1_{n+\tilde{i}\tilde{j}\,2n+3} = -R^2_{\tilde{\tilde{i}}\tilde{j}\,2n+3} = 0, \tag{39}$$

$$R^1_{\tilde{\tilde{i}}\tilde{j}\,2n+4} = R^2_{n+\tilde{i}\tilde{j}\,2n+4} = 0, \quad R^1_{n+\tilde{i}\tilde{j}\,2n+4} = -R^2_{\tilde{\tilde{i}}\tilde{j}\,2n+4} = -\delta_{\tilde{\tilde{i}}\tilde{\tilde{j}}}, \tag{40}$$

$$R^1_{\tilde{\tilde{i}}ab} = R^2_{\tilde{\tilde{i}}ab} = 0, \text{ if } \{a\} \cup \{b\} \not\subset \{\tilde{\tilde{j}}, 2n+3\} \cup \{\tilde{\tilde{j}}, 2n+4\} \text{ for some } \tilde{\tilde{j}}, \tag{41}$$

$$R^1_{\breve{i}\,2n+3\,2n+4} = R^2_{n+\breve{i}\,2n+3\,2n+4} = \sum_{\breve{j}=m+1}^{n} \frac{1}{(N_0+\breve{j}-m-1)!} (x^{2n+2})^{N_0+\breve{j}-m-1}, \tag{42}$$

$$R^1_{n+\breve{i}\,2n+3\,2n+4} = R^2_{\breve{i}\,2n+3\,2n+4} = 0, \quad R^1_{\breve{\breve{i}}ab} = R^2_{\breve{\breve{i}}ab} = 0 \text{ if } \{a\} \cup \{b\} \neq \{2n+3, 2n+4\}. \tag{43}$$

We also wish to have

$$R^1_{\hat{i}ab} = R^2_{n+\hat{i}\,ab} \text{ and } R^1_{n+\hat{i}\,ab} = -R^2_{\hat{i}\,ab}. \tag{44}$$

The computations shows that these equalities hold if we choose

$$\hat{f}_i = f^0_i \quad (i = 1, 2, 3), \tag{45}$$

where the functions f^0_i are as in Section 3.1. In particular,

$$R^1_{\hat{\hat{i}}\hat{\hat{j}}\,2n+4} = \sum_{\alpha=1}^{N} \frac{1}{(\alpha-1)!} A^{\hat{\hat{i}}}_{\alpha\hat{\hat{j}}} (x^{2n+3})^{\alpha-1}. \tag{46}$$

Thus,

$$R^1_{iab} = R^2_{n+i\,ab} \text{ and } R^1_{n+i\,ab} = -R^2_{i\,ab}, \text{ where } 3 \leq i \leq n+2 \tag{47}$$

To compute the covariant derivatives of the curvature tensor we will need the following Christoffel symbols

$$\Gamma^a_{1b} = \Gamma^a_{2b} = 0 \text{ if } a \notin \{1, 2\}, \quad \Gamma^{2n+3}_{ia} = \Gamma^{2n+4}_{ia} = 0, \quad \Gamma^i_{jj_1} = \Gamma^i_{j\,2n+3} = 0, \tag{48}$$

$$\Gamma^{\hat{\hat{i}}}_{\hat{\hat{j}}\,2n+3} = \sum_{\alpha=1}^{N} \frac{1}{\alpha!} A^{\hat{\hat{i}}}_{\alpha\hat{\hat{j}}} (x^{2n+3})^{\alpha}, \quad \Gamma^{\breve{i}}_{\breve{\breve{j}}\,2n+3} = -\Gamma^{\breve{\breve{j}}}_{\breve{i}\,2n+3} = \delta^{\breve{i}+n}_{\breve{\breve{j}}} \hat{\phi}, \tag{49}$$

$$\Gamma^{2n+3}_{2n+3\,2n+3} = -\Gamma^{2n+3}_{2n+4\,2n+4} = \Gamma^{2n+4}_{2n+3\,2n+4} = \hat{\phi},$$
$$-\Gamma^{2n+3}_{2n+3\,2n+4} = \Gamma^{2n+4}_{2n+3\,2n+3} = -\Gamma^{2n+4}_{2n+4\,2n+4} = \hat{\varphi}. \quad (50)$$

Lemma 3.5. *We have*

1) $R^{\hat{\hat{i}}}_{\hat{\hat{j}}ab;a_1;\ldots;a_r} = \sum_{t\in T_{aba_1\ldots a_r}} z_t A^{\hat{\hat{i}}}_{t\hat{\hat{j}}}$, *where $T_{aba_1\ldots a_r}$ is a finite set of indices, z_t are functions and $A_t \in \mathfrak{u}$;*

2) *If $\varphi \neq 0$, then $R^1_{1ab;a_1;\cdots;a_r} = R^2_{2ab;a_1;\cdots;a_r} = \sum_{t\in T_{aba_1\ldots a_r}} z_t\varphi(A_t)$;*

   If $\varphi = 0$, then $R^1_{1ab;a_1;\cdots;a_r} = R^2_{2ab;a_1;\cdots;a_r}$ and this equals 0 for $0 \leq r \leq N-1$;

3) *If $\phi \neq 0$, then $R^2_{1ab;a_1;\cdots;a_r} = -R^1_{2ab;a_1;\cdots;a_r} = \sum_{t\in T_{aba_1\ldots a_r}} z_t\phi(A_t)$, $R^{\check{i}}_{\check{\check{j}}ab;a_1;\cdots;a_r} = -R^{\check{\check{j}}}_{\check{i}ab;a_1;\cdots;a_r} = \delta^{\check{i}+n}_{\check{\check{j}}} \sum_{t\in T_{aba_1\ldots a_r}} z_t\phi(A_t)$;*

   If $\phi = 0$, then $R^2_{1ab;a_1;\cdots;a_r} = -R^1_{2ab;a_1;\cdots;a_r}$, $R^{\check{i}}_{\check{\check{j}}ab;a_1;\cdots;a_r} = -R^{\check{\check{j}}}_{\check{i}ab;a_1;\cdots;a_r}$ and these components equals 0 for $0 \leq r \leq N-1$;

4) *$R^1_{iab;a_1;\ldots;a_r} = R^2_{n+i\,ab;a_1;\ldots;a_r}$ and $R^1_{n+i\,ab;a_1;\ldots;a_r} = -R^2_{i\,ab;a_1;\ldots;a_r}$, where $3 \leq i \leq n+2$;*

5) *$R^1_{n+\check{i}\,ab;a_1;\ldots;a_r} = R^2_{\check{i}ab;a_1;\ldots;a_r} = 0$;*

6) *$R^i_{jab;a_1;\ldots;a_r} = 0$ if $i,j \notin \{3,...,n_0+2, n+3,...,n+n_0+2\}$ or $i,j \notin \{m+3,...,n+2, n+m+3,...,2n+2\}$.*

Proof. The lemma can be easily proved using the induction and equalities (27), (28), (35–38), (43) and (47).

For example, let us prove Part 1) and Part 2) of the lemma for $a_r = 2n+4$. Let $r \geq 1$ Suppose that the lemma is true for all $s < r$. Let $a_r = 2n+4$. Suppose that $\varphi \neq 0$.

We have

$R^i_{jbc;a_1;\ldots;a_{r-1};2n+4} =$

$$\frac{\partial R^i_{jbc;a_1;\ldots;a_{r-1}}}{\partial x^{2n+4}} + \Gamma^i_{a\,2n+4}R^a_{jbc;a_1;\ldots;a_{r-1}} - \Gamma^a_{j\,2n+4}R^i_{abc;a_1;\ldots;a_{r-1}} - \Gamma^a_{b\,2n+4}R^i_{jac;a_1;\ldots;a_{r-1}}$$
$$-\Gamma^a_{c\,2n+4}R^i_{jba;a_1;\ldots;a_{r-1}} - \Gamma^a_{a_1\,2n+4}R^i_{jbc;a;a_2\ldots;a_{r-1}} - \ldots - \Gamma^a_{a_{r-1}\,2n+4}R^i_{jbc;a_1;\ldots;a_{r-2};a}.$$

Since $\mathfrak{hol}_0 \subset \mathfrak{so}(2,2n+2)_{<p_1,p_2>}$, we have $R^{2n+3}_{jbc;a_1;\ldots;a_{r-1}} = R^{2n+4}_{jbc;a_1;\ldots;a_{r-1}} = R^i_{1bc;a_1;\ldots;a_{r-1}} = R^i_{2bc;a_1;\ldots;a_{r-1}} = 0$. Using this, (48), (49) and the induction hypotheses, we get

$R^i_{jbc;a_1;\dots;a_{r-1};2n+4} =$

$$\sum_{t\in T_{bca_1\dots a_{r-1}}} \frac{\partial z_t}{\partial x^{2n+4}} A^i_{tj} + \sum_{\alpha=1}^{N} \frac{1}{\alpha!} \sum_{t\in T_{bca_1\dots a_r}} (x^{2n+3})^\alpha z_t [A_\alpha, A_t]^i_j$$
$$- \sum_{a=1}^{2n+4} \sum_{t\in T_{ac;a_1;\dots;a_{r-1}}} \Gamma^a_{b\,2n+4} z_t A^i_{tj} - \sum_{a=1}^{2n+4} \sum_{t\in T_{ba;a_1;\dots;a_{r-1}}} \Gamma^a_{c\,2n+4} z_t A^i_{tj}$$
$$- \sum_{a=1}^{2n+4} \sum_{t\in T_{bc;a;a_2;\dots;a_{r-1}}} \Gamma^a_{a_1\,2n+4} z_t A^i_{tj} - \dots - \sum_{a=1}^{2n+4} \sum_{t\in T_{bc;a_1;\dots;a_{r-2};a}} \Gamma^a_{a_{r-1}\,2n+4} z_t A^i_{tj}.$$

We also have

$R^1_{1bc;a_1;\dots;a_{r-1};2n+4} =$

$$\frac{\partial R^1_{1bc;a_1;\dots;a_{r-1}}}{\partial x^{2n+4}} + \Gamma^1_{a\,2n+4} R^a_{1bc;a_1;\dots;a_{r-1}} - \Gamma^a_{1\,2n+4} R^1_{abc;a_1;\dots;a_{r-1}} - \Gamma^a_{b\,2n+4} R^1_{1ac;a_1;\dots;a_{r-1}}$$
$$- \Gamma^a_{c\,2n+4} R^1_{1ba;a_1;\dots;a_{r-1}} - \Gamma^a_{a_1\,2n+4} R^1_{1bc;a;a_2\dots;a_{r-1}} - \dots - \Gamma^a_{a_{r-1}\,2n+4} R^1_{1bc;a_1;\dots;a_{r-2};a}.$$

Since $\mathfrak{hol}_0\mathfrak{so}(2,2n+2)_{<p_1,p_2>}$, we see that $R^a_{1bc;a_1;\dots;a_{r-1}} = \Gamma^a_{1\,2n+4} = 0$ if $a \not\in \{1,2\}$. Hence,

$$\begin{aligned}&\Gamma^1_{a\,2n+4} R^a_{1bc;a_1;\dots;a_{r-1}} - \Gamma^a_{1\,2n+4} R^1_{abc;a_1;\dots;a_{r-1}}\\ &= \Gamma^1_{2\,2n+4} R^2_{1bc;a_1;\dots;a_{r-1}} - \Gamma^2_{1\,2n+4} R^1_{2bc;a_1;\dots;a_{r-1}}\\ &= (\Gamma^1_{2\,2n+4} + \Gamma^2_{1\,2n+4}) R^2_{1bc;a_1;\dots;a_{r-1}} = 0,\end{aligned}$$

where we used Statement 3) for $r-1$ and the fact that $\Gamma^1_{2\,2n+4} + \Gamma^2_{1\,2n+4} = 0$. Thus,

$R^1_{1bc;a_1;\dots;a_{r-1};2n+4} =$

$$\sum_{t\in T_{bca_1\dots a_{r-1}}} \frac{\partial z_t}{\partial x^{2n+4}} \varphi(A_t)$$
$$- \sum_{a=1}^{2n+4} \sum_{t\in T_{ac;a_1;\dots;a_{r-1}}} \Gamma^a_{b\,2n+4} z_t \varphi(A_t) - \sum_{a=1}^{2n+4} \sum_{t\in T_{ba;a_1;\dots;a_{r-1}}} \Gamma^a_{c\,2n+4} z_t \varphi(A_t)$$
$$- \sum_{a=1}^{2n+4} \sum_{t\in T_{bc;a;a_2;\dots;a_{r-1}}} \Gamma^a_{a_1\,2n+4} z_t \varphi(A_t) - \dots - \sum_{a=1}^{2n+4} \sum_{t\in T_{bc;a_1;\dots;a_{r-2};a}} \Gamma^a_{a_{r-1}\,2n+4} z_t \varphi(A_t).$$

In the same way we can compute $R^2_{2bc;a_1;\dots;a_{r-1};2n+4}$. Now the statement follows from the induction hypotheses and the fact that $\varphi|_{\mathfrak{u}'} = 0$. $\Box$

Lemma 3.6. *If one of the numbers a, b, a_1,...,a_r belongs to the set $\{1,\dots,2n+2\}$, then*

$$R^i_{jab;a_1;\dots;a_r} = 0.$$

Proof. To prove the lemma it is enough to prove the following 3 statements

1) If $1 \le c \le 2n+2$, then $\frac{\partial R^i_{jab;a_1;\dots;a_{r-1}}}{\partial x^c} = 0$;

2) If $1 \le c \le 2n+2$, then $R^i_{jab;a_1;\dots;a_{r-1};c} = 0$;

3) If for fixed a, b, a_1,...,a_{r-1} (and for all $1 \le i,j \le j$) we have $R^i_{jab;a_1;\dots;a_{r-1}} = 0$, then $R^i_{jab;a_1;\dots;a_{r-1};a_r} = 0$. These statements can be proved using the induction, (35) and (48). $\Box$

Lemma 3.7. *For all $0 \le r \le N-1$ we have*

$$R^{\hat{\hat{i}}}_{\hat{\hat{j}}\,2n+3\,2n+4;2n+3;\ldots;2n+3(r\ \text{times})}(0) = A^{\hat{\hat{i}}}_{r+1\,\hat{\hat{j}}} + \sum_{\alpha=1}^{r} \mu_{r\alpha} A^{\hat{\hat{i}}}_{\alpha\hat{\hat{j}}},$$

where $\mu_{r\alpha}$ are some numbers.

Proof. We will prove the following three statements

1) If $0 \le r \le N_1 - 1$, then

$$R^{\hat{\hat{i}}}_{\hat{\hat{j}}\,2n+3\,2n+4;2n+3;\ldots;2n+3(r\ \text{times})} = \sum_{\alpha=r+1}^{N} \frac{1}{(\alpha-r-1)!} A^{\hat{\hat{i}}}_{\alpha\hat{\hat{j}}}(x^{2n+3})^{\alpha-r-1} + \sum_{\beta=1}^{N} H_\beta(x^{2n+3}) A^{\hat{\hat{i}}}_{\beta\hat{\hat{j}}},$$

where $H_\beta(x^{2n+3})$ are polynomials of x^{2n+3} such that $H_\beta(0) = H'_\beta(0) = \cdots = H^{(N_1-r)}_\beta(0) = 0$;

2) For all $0 \le r \le N-1$ we have

$$R^{\hat{\hat{i}}}_{\hat{\hat{j}}\,2n+3\,2n+4;a_1;\ldots;a_r} = \sum_{\alpha=1}^{r+1} G_\alpha(x^{2n+3}) A^{\hat{\hat{i}}}_{\alpha\hat{\hat{j}}} + \sum_{\alpha=r+2}^{N} F_\alpha(x^{2n+3}) A^{\hat{\hat{i}}}_{\alpha\hat{\hat{j}}},$$

where $G_\alpha(x^{2n+3})$ and $F_\alpha(x^{2n+3})$ are polynomials of x^{2n+3} such that $F_{r+2}(0) = 0$, $F_{r+3}(0) = F'_{r+3}(0) = 0$,...,$F_N(0) = F'_N(0) = \cdots = F_N^{(N-r-2)}(0) = 0$;

3) If $N_1 \le r \le N-1$, then

$$R^{\hat{\hat{i}}}_{\hat{\hat{j}}\,2n+3\,2n+4;2n+3;\ldots;2n+3(r\ \text{times})} =$$

$$\sum_{\alpha=r+1}^{N} \frac{1}{(\alpha-r-1)!} A^{\hat{\hat{i}}}_{\alpha\hat{\hat{j}}}(x^{2n+3})^{\alpha-r-1} + \sum_{\beta=1}^{N_1} W_\beta(x^{2n+3}) A^{\hat{\hat{i}}}_{\beta\hat{\hat{j}}} + \sum_{\gamma=N_1+1}^{N} Q_\gamma(x^{2n+3}) A^{\hat{\hat{i}}}_{\gamma\hat{\hat{j}}},$$

where $W_\beta(x^{2n+3})$ and $Q_\gamma(x^{2n+3})$ polynomials of x^{2n+3} such that $Q_{r+1}(0) = 0$, $Q_{r+2}(0) = Q'_{r+2}(0) = 0$,...,$Q_N(0) = Q'_N(0) = \cdots = Q_N^{(N-r-1)}(0) = 0$.

Let us prove Statement 1). For $r = 0$ the statement follows from (35). Let $r > 0$ and suppose that Statement 1) holds for all $s < r$. We have

$$R^{\hat{\hat{i}}}_{\hat{\hat{j}}\,2n+3\,2n+4;2n+3;\ldots;2n+3(r\ \text{times})} =$$

$$\frac{\partial R^{\hat{\hat{i}}}_{\hat{\hat{j}}\,2n+3\,2n+4;2n+3;\ldots;2n+3(r-1\ \text{times})}}{\partial x^{2n+3}} + \Gamma^{\hat{\hat{i}}}_{a\,2n+3} R^{a}_{\hat{\hat{j}}\,2n+3\,2n+4;2n+3;\ldots;2n+3(r-1\ \text{times})}$$

$$-\Gamma^{a}_{\hat{\hat{j}}\,2n+3} R^{\hat{\hat{i}}}_{a\,2n+3\,2n+4;2n+3;\ldots;2n+3(r-1\ \text{times})}$$

$$-\Gamma^{a}_{2n+3\,2n+3} R^{\hat{\hat{i}}}_{\hat{\hat{j}}\,a\,2n+4;2n+3;\ldots;2n+3(r-1\ \text{times})} - \Gamma^{a}_{2n+4\,2n+3} R^{\hat{\hat{i}}}_{\hat{\hat{j}}\,2n+3\,a;2n+3;\ldots;2n+3(r-1\ \text{times})}$$

$$-\Gamma^{a}_{2n+3\,2n+3} R^{\hat{\hat{i}}}_{\hat{\hat{j}}\,2n+3\,2n+4;a;2n+3;\ldots;2n+3(r-2\ \text{times})}$$

$$-\cdots - \Gamma^{a}_{2n+3\,2n+3} R^{\hat{\hat{i}}}_{\hat{\hat{j}}\,2n+3\,2n+4;2n+3;\ldots;2n+3(r-2\ \text{times});a}.$$

Using Lemma 3.6, (48) and (50) we get

$$
\begin{aligned}
&R^{\hat{\hat{i}}}_{\hat{\hat{j}}\,2n+3\,2n+4;2n+3;\dots;2n+3(r\text{ times})} = \\
&\frac{\partial R^{\hat{\hat{i}}}_{\hat{\hat{j}}\,2n+3\,2n+4;2n+3;\dots;2n+3(r-1\text{ times})}}{\partial x^{2n+3}} - (r+1)\hat{\phi}(x^{2n+3})R^{\hat{\hat{i}}}_{\hat{\hat{j}}\,2n+3\,2n+4;2n+3;\dots;2n+3(r-1\text{ times})} \\
&-\hat{\varphi}(x^{2n+3})(R^{\hat{\hat{i}}}_{\hat{\hat{j}}\,2n+3\,2n+4;2n+4;2n+3;\dots;2n+3(r-2\text{ times})} \\
&+\dots+R^{\hat{\hat{i}}}_{\hat{\hat{j}}\,2n+3\,2n+4;2n+3;\dots;2n+3(r-2\text{ times});2n+4}).
\end{aligned}
\tag{51}
$$

Statement 1) follows from the induction hypotheses and the fact that $\hat{\varphi}(0)=\hat{\varphi}'(0)=\dots=\hat{\varphi}^{(N_1-1)}(0)=0$.

Statement 2) can by proved by analogy (by Lemma 3.6 we may consider only $a_r=2n+3$ and $2n+4$). Statement 3) can be proved using (51) and Statement 2).

From Statement 1) it follows that for $0\le r\le N_1-1$ we have

$$R^{\hat{\hat{i}}}_{\hat{\hat{j}}\,2n+3\,2n+4;2n+3;\dots;2n+3(r\text{ times})}(0)=A^{\hat{\hat{i}}}_{r+1\,\hat{\hat{j}}}.$$

The end of the proof of the lemma follows from this and Statement 3). $\square$

From Statement 1) of Lemma 3.5 and Lemma 3.7 it follows that $\mathrm{pr}_{\mathfrak{so}(2m)}\,\mathfrak{hol}_0=\mathfrak{u}$.

Lemma 3.8. *For all* $0\le r\le N-1$ *we have*

$$R^{1}_{\hat{\hat{i}}\hat{\hat{j}}\,2n+4;2n+3;\dots;2n+3(r\ \textit{times})}(0)=A^{\hat{\hat{i}}}_{r+1\,\hat{\hat{j}}}+\sum_{\alpha=1}^{r}\nu_{r\alpha}A^{\hat{\hat{i}}}_{\alpha\hat{\hat{j}}},$$

where $\nu_{r\alpha}$ *are some numbers.*

Proof. The lemma can be proved by analogy to the proof of Lemma 3.7 using (46). $\square$

Using (42), we get

$$
\begin{aligned}
R^{1}_{\check{i}\,2n+3\,2n+4;2n+3;\dots;2n+3(r\text{ times})}(0) &= R^{2}_{n+\check{i}\,2n+3\,2n+4;2n+3;\dots;2n+3(r\text{ times})}(0)\\
&=\delta_{\check{i}\,r-N_0+m+3}\ \text{for all } N_0\le r\le N_0+n-m-1.
\end{aligned}
\tag{52}
$$

Consider the Lie algebra $\mathfrak{hol}^{m,\mathfrak{u},\mathcal{A}^1,\tilde{A}^2}$. We take $\varphi=\phi=0$, $N_0=N+2$, $\varphi_{N+1}=1$, $\varphi_{N+2}=0$, $\phi_{N+1}=0$ and $\phi_{N+2}=1$. Then the above metric coincides with the metric from Table 3.2 for the Lie algebra $\mathfrak{hol}^{m,\mathfrak{u},\mathcal{A}^1,\tilde{A}^2}$. From Lemma 3.5 it follows that $\mathfrak{hol}_0\subset\mathfrak{hol}^{m,\mathfrak{u},\mathcal{A}^1,\tilde{A}^2}$. From Lemma 3.7 it follows that $\mathfrak{u}\subset\mathfrak{hol}_0$. From (39), (40), (52), Lemma 3.8 and the fact that $\mathfrak{u}$ does not annihilate any proper subspace of $E_{1,\dots,n_0}$ it follows that $\mathcal{N}^1+\mathcal{N}^2\subset\mathfrak{hol}_0$. It can be shown that

$$R^{1}_{1\,2n+3\,2n+4;2n+3;\dots;2n+3(N+1\text{ times})}(0)=R^{2}_{2\,2n+3\,2n+4;2n+3;\dots;2n+3(N+1\text{ times})}(0)=1,$$

$$R^{2}_{1\,2n+3\,2n+4;2n+3;\dots;2n+3(N+1\text{ times})}(0)=R^{1}_{2\,2n+3\,2n+4;2n+3;\dots;2n+3(N+1\text{ times})}(0)=0,$$

$$R^{1}_{1\,2n+3\,2n+4;2n+3;\dots;2n+3(N+2\text{ times})}(0)=R^{2}_{2\,2n+3\,2n+4;2n+3;\dots;2n+3(N+2\text{ times})}(0)=0,$$

$$R^{2}_{1\,2n+3\,2n+4;2n+3;\dots;2n+3(N+2\text{ times})}(0)=-R^{1}_{2\,2n+3\,2n+4;2n+3;\dots;2n+3(N+2\text{ times})}(0)=1,$$

$$R^{\breve{i}}_{\breve{\breve{j}}\,2n+3\,2n+4;2n+3;\dots;2n+3(N+2\text{ times})}(0)=-R^{\breve{\breve{j}}}_{\breve{i}\,2n+3\,2n+4;2n+3;\dots;2n+3(N+2\text{ times})}(0)=\delta^{\breve{i}+n}_{\breve{\breve{j}}}.$$

Hence, $\mathcal{A}^1+\tilde{\mathcal{A}}^2\subset\mathfrak{hol}_0$. The inclusion $\mathcal{C}\subset\mathfrak{hol}_0$ is obvious. Thus, $\mathfrak{hol}_0=\mathfrak{hol}^{n,\mathfrak{u},\mathcal{A}^1,\tilde{\mathcal{A}}^2}$.

The Lie algebras $\mathfrak{hol}^{m,\mathfrak{u},\mathcal{A}^1,\phi}$, $\mathfrak{hol}^{m,\mathfrak{u},\varphi,\tilde{\mathcal{A}}^2}$, $\mathfrak{hol}^{m,\mathfrak{u},\varphi,\phi}$ and $\mathfrak{hol}^{m,\mathfrak{u},\lambda}$, can be considered in the same way.

Now we are left with the Lie algebras $\mathfrak{hol}^{n,\mathfrak{u},\psi,k,l}$ and $\mathfrak{hol}^{m,\mathfrak{u},\psi,k,l,r}$. For them we can use the following lemma. Let $\bar{\imath}=n_0+3,\dots,n+2,n+n_0+3,\dots,2n+2$.

Lemma 3.9. *Consider the following metrics on* $\mathbb{R}^{2n+4}$

$$\begin{aligned}
g=\;&2dx^1dx^{2n+3}+2dx^2dx^{2n+4}+\textstyle\sum_{i=3}^{2n+2}(dx^i)^2+2\sum_{i=3}^{2n+2}u^i(x^{\hat{\hat{i}}},x^{2n+3})dx^idx^{2n+4}\\
&+(\hat{f}_1(x^{\hat{\hat{i}}},x^{2n+3})+\bar{f}_1(x^{\bar{i}},x^{2n+3}))(dx^{2n+3})^2+(\hat{f}_2(x^{\hat{\hat{i}}},x^{2n+3})+\bar{f}_2(x^{\bar{i}},x^{2n+3}))(dx^{2n+4})^2\\
&+2(\hat{f}_3(x^{\hat{\hat{i}}},x^{2n+3})+\bar{f}_3(x^{\bar{i}},x^{2n+3}))dx^{2n+3}dx^{2n+4},\\
g_1=\;&2dx^1dx^{2n+3}+2dx^2dx^{2n+4}+\textstyle\sum_{i=3}^{2n+2}(dx^i)^2+2\sum_{i=3}^{2n+2}u^i(x^{\hat{\hat{i}}},x^{2n+3})dx^idx^{2n+4}\\
&+\hat{f}_1(x^{\hat{\hat{i}}},x^{2n+3})(dx^{2n+3})^2+\hat{f}_2(x^{\hat{\hat{i}}},x^{2n+3})(dx^{2n+4})^2+2\hat{f}_3(x^{\hat{\hat{i}}},x^{2n+3})dx^{2n+3}dx^{2n+4},\\
g_2=\;&2dx^1dx^{2n+3}+2dx^2dx^{2n+4}+\textstyle\sum_{i=3}^{2n+2}(dx^i)^2\\
&+\bar{f}_1(x^{\bar{i}},x^{2n+3})(dx^{2n+3})^2+\bar{f}_2(x^{\bar{i}},x^{2n+3})(dx^{2n+4})^2+2\bar{f}_3(x^{\bar{i}},x^{2n+3})dx^{2n+3}dx^{2n+4}.
\end{aligned}$$

Let R, $\hat{R}$ and $\bar{R}$ be the corresponding curvature tensors, then $R=\hat{R}+\bar{R}$.

Proof. Using equalities (16) – (26), it is easy to see that for the corresponding Christoffel symbols we have $\Gamma^a_{bc}=\hat{\Gamma}^a_{bc}+\bar{\Gamma}^a_{bc}$, $\hat{\Gamma}^a_{bc}\bar{\Gamma}^{a_1}_{ab_1}=0$ and $\bar{\Gamma}^a_{bc}\hat{\Gamma}^{a_1}_{ab_1}=0$. The proof of the lemma follows from the formula for the curvature tensor. $\square$

Consider the Lie algebra $\mathfrak{hol}^{n,\mathfrak{u},\psi,k,l}$. We have $f_i=f_i^0+\tilde{f}^k_{i\,n_0+1}+f_i^{n,\psi}+\breve{f}^{N+1\,n}_{i\,l+1}$ $(i=1,2,3)$. From (16) and (19) it follows that the holonomy algebra $\mathfrak{hol}_0$ annihilats the vectors p_1 and p_2.

By Lemma 3.9, it is enough to compute the covariant derivatives of the curvature tensor $\hat{R}$ of the metric g_1 with $f_i=f_i^0+\tilde{f}^k_{i\,n_0+1}$ $(i=1,2,3)$ and the covariant derivatives of the curvature tensor $\bar{R}$ of the metric g_2 with $f_i=f_i^{n,\psi}+\breve{f}^{N+1\,n}_{i\,l+1}$ $(i=1,2,3)$.

Consider the curvature tensor $\hat{R}$. Set $\varphi=\phi=0$ and $N_0=N$, then we can use the above computations.

As in Lemma 3.7, we can show that for all $0 \leq r \leq N-1$ it holds

$$\hat{R}^i_{j\,2n+3\,2n+4;2n+3;\ldots;2n+3(r \text{ times})}(0) = A^i_{r+1\,j},$$

and $\hat{R}^i_{j\,2n+3\,2n+4;2n+3;\ldots;2n+3(r \text{ times})}(0) = 0$ for $r \geq N$.

From (50) and proof of Lemma 3.5 it follows that

$$\hat{R}^i_{jbc;a_1;\ldots;a_{r-1};2n+4} = \sum_{\alpha=1}^{N} \frac{1}{\alpha!}(x^{2n+3})^\alpha [A_\alpha, \hat{R}(b,c;a_1;\ldots;a_{r-1})]^i_j.$$

Hence if at least one of the numbers $a_1, \ldots, a_r$ equals $2n+4$, then $\hat{R}(b,c;a_1;\ldots;a_r) \in \mathfrak{u}'$. We can use also Lemma 3.6, (39) and (40).

For the curvature tensor $\bar{R}$ we have $\bar{R}^i_{jbc;a_1;\ldots;a_r} = 0$. Let $3 \leq i \leq n+2$, then

$$\begin{aligned}
\bar{R}^1_{i\,2n+3\,2n+4;2n+3;\ldots;2n+3(r \text{ times})}(0) &= \bar{R}^2_{i+n\,2n+3\,2n+4;2n+3;\ldots;2n+3(r \text{ times})}(0) \\
&= \begin{cases} -\psi_{1\,r+1\,i}, & \text{if } k+3 \leq i \leq l+2 \text{ and } N_1 \leq r \leq N-1, \\ 1, & \text{if } l+3 \leq i \leq n+2 \text{ and } r = N+i-l-3, \\ 0, & \text{else,} \end{cases}
\end{aligned}$$

$$\begin{aligned}
\bar{R}^1_{i+n\,2n+3\,2n+4;2n+3;\ldots;2n+3(r \text{ times})}(0) &= -\bar{R}^2_{i\,2n+3\,2n+4;2n+3;\ldots;2n+3(r \text{ times})}(0) \\
&= \begin{cases} \psi_{2\,r+1\,i}, & \text{if } k+3 \leq i \leq l+2 \text{ and } N_1 \leq r \leq N-1, \\ \psi_{3\,r+1\,i}, & \text{if } l+3 \leq i \leq n+2 \text{ and } N_1 \leq r \leq N-1, \\ 0, & \text{else.} \end{cases}
\end{aligned}$$

It can be also proved that $\bar{R}^1_{jbc;a_1;\ldots;a_r} = \bar{R}^2_{jbc;a_1;\ldots;a_r} = 0$, if $\{b\} \cup \{c\} \neq \{2n+3, 2n+4\}$ or $\{a_1\} \cup \cdots \cup \{a_r\} \neq \{2n+3\}$.

Now it is easy to see that $\mathfrak{hol}_0 = \mathfrak{hol}^{n,\mathfrak{u},\psi,k,l}$. The Lie algebra $\mathfrak{hol}^{m,\mathfrak{u},\psi,k,l,r}$ can be considered in the same way.

The theorem is proved. $\Box$.

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